A Mosco Stability Theorem for the Generalized Proximal Mapping

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ABSTRACT. We consider the generalized proximal mapping $\mathrm{Prox}_{\varphi}^{f} := (\nabla f + \partial\varphi)^{-1}$ in which f is a Legendre function and φ is a proper, lower semicontinuous, convex function on a reflexive Banach space X. Does the sequence $\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)$ converge weakly or strongly to $\mathrm{Prox}_{\varphi}^{f}(\xi)$ as the functions φ_n Mosco-converge to φ and the vectors ξ_n converge to $\xi \in \operatorname{int}\operatorname{dom} f^*$? Previous results show that, if the functions φ_n are uniformly bounded from below, then weak convergence holds when f is strongly coercive or uniformly convex on bounded sets, with strong convergence resulting from weak convergence whenever f is totally convex. We prove that the same is true when f is only coercive and the sequence $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above. In this context, we establish some continuity type properties of $\mathrm{Prox}_{\varphi}^{f}$.

1. Introduction

In this paper X denotes a real reflexive Banach space with the norm $\|\cdot\|$ and X^* represents the (topological) dual of X whose norm is denoted $\|\cdot\|_*$. Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous, convex function and let $f^* : X^* \to (-\infty, +\infty]$ be the Fenchel conjugate of f. *All over this paper we assume that f is a Legendre function* (see [**8**, Definition 5.2]).

1.1 Some facts about Legendre functions. Recall that, according to [**8**, Theorems 5.4 and 5.6], the function f is Legendre if and only if it satisfies the following conditions:

(L1) The interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is differentiable on $\operatorname{int}\operatorname{dom} f$ and

$$\operatorname{dom}\partial f = \operatorname{int}\operatorname{dom} f; \tag{1.1}$$

(L2) The interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$, is nonempty, f^* is differentiable on $\operatorname{int}\operatorname{dom} f^*$ and

$$\operatorname{dom}\partial f^* = \operatorname{int}\operatorname{dom} f^*. \tag{1.2}$$

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Since X is reflexive we also have $(\partial f)^{-1} = \partial f^*$ (see [**13**, p. 83]). This fact, combined with conditions (L1) and (L2), implies the following equalities which we use in the sequel:

$$\nabla f = (\nabla f^*)^{-1}, \tag{1.3}$$

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*, \tag{1.4}$$

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f. \tag{1.5}$$

Also, conditions (L1) and (L2) in conjunction with [**8**, Theorem 5.4] imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Several interesting examples of Legendre functions are presented in [**7**] and [**8**]. Among them are the functions $\frac{1}{s}\|\cdot\|^s$ with $s \in (1, \infty)$ when the space X is smooth and strictly convex and, in particular, when X is a Hilbert space.

1.2 The proximal mapping relative to f. We denote by $\mathcal{F}_f$ the set of proper, lower semicontinuous, convex functions $\varphi : X \to (-\infty, +\infty]$ which satisfy the conditions that

$$\operatorname{dom}\varphi \cap \operatorname{int}\operatorname{dom} f \neq \varnothing, \tag{1.6}$$

and

$$\varphi_f := \inf\{\varphi(x) : x \in \operatorname{dom}\varphi \cap \operatorname{dom} f\} > -\infty. \tag{1.7}$$

According to [**9**, Propositions 3.22 and 3.23] (see [**17**, Lemma 2.1] for another proof of the same result), for any $\varphi \in \mathcal{F}_f$, the operator $\operatorname{Prox}_\varphi^f : X^* \to 2^X$ given by

$$\operatorname{Prox}_\varphi^f(\xi) := \arg\min\left\{\varphi(x) + W^f(\xi, x) : x \in X\right\}, \tag{1.8}$$

where

$$W^f(\xi, x) := f(x) - \langle \xi, x\rangle + f^*(\xi), \tag{1.9}$$

is single valued on $\operatorname{int}\operatorname{dom} f^*$ and, for any $\xi \in \operatorname{int}\operatorname{dom} f^*$, it has

$$\operatorname{Prox}_\varphi^f(\xi) == (\partial\varphi + \nabla f)^{-1}(\xi) = [\partial(\varphi + f)]^{-1}(\xi), \tag{1.10}$$

and

$$\operatorname{Prox}_\varphi^f(\xi) \in \operatorname{dom}\partial\varphi \cap \operatorname{int}\operatorname{dom} f. \tag{1.11}$$

We call $\operatorname{Prox}_\varphi^f$ *the (generalized) proximal mapping relative to* f *associated to* φ. Denote

$$\operatorname{Env}_\varphi^f(\xi) = \inf\{\varphi(x) + W^f(\xi, x) : x \in X\}. \tag{1.12}$$

Then, for each $\xi \in \operatorname{int}\operatorname{dom} f^*$, the vector $\operatorname{Prox}_\varphi^f(\xi)$ is the only vector in X such that

$$\operatorname{Env}_\varphi^f(\xi) = \varphi(\operatorname{Prox}_\varphi^f(\xi)) + W^f(\xi, \operatorname{Prox}_\varphi^f(\xi)). \tag{1.13}$$

The notions of proximal mapping relative to f, $\operatorname{Prox}_\cdot^f$, and of envelope, $\operatorname{Env}_\cdot^f$, are natural generalizations of the classical concepts of proximal mapping and envelope originally introduced and studied in Hilbert spaces for $f = \frac{1}{2}\|\cdot\|^2$ by Moreau [**25**], [**26**], [**27**] and Rockafellar [**34**], [**35**] (see [**36**] for more historical comments on this topic) and subsequently placed in a more general context in works like [**14**] and [**29**]. To the best of our knowledge, the generalizations defined by (1.8) and (1.12) go back to Alber's works [**1**] and [**2**].

1.3 Mosco stability of the proximal mapping: The concept. In this paper we are concerned with the question whether the operator $\mathrm{Prox}_{\varphi}^{f}$ is *stable* with respect to the Mosco convergence. Precisely, we would like to know whether, and under which conditions, given the functions $\varphi_n, \varphi : X \to (-\infty, +\infty]$, $(n \in \mathbb{N})$, contained in $\mathcal{F}_f$ and such that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to φ, and given a convergent sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom} f^*$ with $\lim_{n\to\infty} \xi_n = \xi \in \operatorname{int}\operatorname{dom} f^*$, does the sequence $\left\{\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)\right\}_{n\in\mathbb{N}}$ converges (weakly or strongly) to $\mathrm{Prox}_{\varphi}^{f}(\xi)$? Recall (see [**28**, Definition 1.1 and Lemma 1.10]) that the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to be *convergent in the sense of Mosco to* φ (and we write $\text{M-}\lim_{n\to\infty} \varphi_n = \varphi$) if the following conditions are satisfied:

(M1) If $\{x_n\}_{n\in\mathbb{N}}$ is a weakly convergent sequence in X such that $\text{w-}\lim_{n\to\infty} x_n = x$, and if $\{\varphi_{i_n}\}_{n\in\mathbb{N}}$ is a subsequence $\{\varphi_n\}_{n\in\mathbb{N}}$, then $\liminf_{n\to\infty} \varphi_{i_n}(x_n) \geq \varphi(x)$;

(M2) For every $u \in X$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty} \varphi_n(u_n) = \varphi(u). \tag{1.14}$$

Stability properties with respect to Mosco convergence of the proximal mapping $\mathrm{Prox}_{\varphi}^{f}$ are already known to hold in various circumstances similar to those described here. For instance, Theorem 3.26 in [**6**] implies that if X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then $\mathrm{Prox}_{\varphi_n}^{f}(\xi)$ converges strongly to $\mathrm{Prox}_{\varphi}^{f}(\xi)$ whenever $\text{M-}\lim_{n\to\infty} \varphi_n = \varphi$ and $\xi \in X^*$. Generalizations of this result occur in [**18**], [**22**], [**23**], [**32**] and they are summarized in [**19**] as corollaries of Theorem 2.1 there. Theorem 2.1 of [**19**] shows that if the Legendre function f is either strongly coercive (i.e., $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$) or uniformly convex on bounded sets (see [**37**]), if the functions $\varphi_n, \varphi : X \to (-\infty, +\infty]$, $(n \in \mathbb{N})$, contained in $\mathcal{F}_f$, are uniformly bounded from below and $\text{M-}\lim_{n\to\infty} \varphi_n = \varphi$, then $\mathrm{Prox}_{\varphi}^{f}$ is weakly stable (i.e. $\mathrm{Prox}_{\varphi}^{f}(\xi_n)$ converges weakly to $\mathrm{Prox}_{\varphi}^{f}(\xi)$ whenever $\{\xi_n\}_{n\in\mathbb{N}} \subset \operatorname{int}\operatorname{dom} f^*$ has $\lim_{n\to\infty} \xi_n = \xi \in \operatorname{int}\operatorname{dom} f^*$). Moreover, it also results from Theorem 2.1 in [**19**] that whenever weak stability of $\mathrm{Prox}_{\varphi}^{f}$ can be ensured and the Legendre function f is totally convex, then the convergence of $\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)$ to $\mathrm{Prox}_{\varphi}^{f}(\xi)$ is strong, that is, strong stability holds.

Recall (cf. [**15**]) that the function f is called totally convex if, for each $x \in \operatorname{int}\operatorname{dom} f$, the modulus of total convexity of f at x which is defined by

$$\nu_f(x,t) = \inf\left\{W^f(\nabla f(x), y) : \|y - x\| = t\right\}$$

is positive whenever $t > 0$. Total convexity is a common feature of a pletora of Legendre functions in reflexive Banach spaces. For example, if X is smooth, strictly convex and has the Kadec-Klee property then all functions $\frac{1}{s}\|\cdot\|^s$ with $s \in (1, \infty)$, are totally convex Legendre functions (cf. [**18**, Section 2.3]). In particular, this happens when X is uniformly smooth and uniformly convex as many usual spaces (like Hilbert spaces, Lebesgue spaces, Sobolev spaces) are.

The relevance of the results concerning the Mosco stability of the proximal mapping with functions f which are not necessarily the square of the norm should be seen in the larger context of the analysis of generalized variational inequalities requiring to find $x \in \operatorname{int}\operatorname{dom} f$ such that

$$\exists\, \xi \in Bx : [\langle \xi, y - x\rangle \geq \varphi(x) - \varphi(y),\ \forall y \in \operatorname{dom} f]\,, \tag{1.15}$$

where $\varphi \in \mathcal{F}_f$ and $B : X \to 2^{X^*}$ is an operator which satisfies some conditions (see, for instance, [**5**], [**13**], [**17**], [**20**] and [**28**] for more details on this topic). Mosco stability is a tool of ensuring that, in some circumstances, "small" data perturbations in (1.15) do not essentially alter its solution. The main result in [**19**], described above, involving the requirement of uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$, naturally applies to classical variational inequalities where the function φ and its perturbations φ_n usually are indicator functions of closed convex sets. However, the uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$ happens to be a restrictive condition for the study of some non-classical generalized variational inequalities.

This leads us to the topic of the current paper. Can stability with respect to Mosco convergence of the proximal mapping be established in conditions which are different and, hopefully, less demanding than those mentioned above? That uniform boundedness from below of $\{\varphi_n\}_{n\in\mathbb{N}}$ (as presumed in [**19**]) is not a necessary condition for the weak/strong convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$ can be observed from [**6**, Theorem 3.26] which applies in our setting when X is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$. Our main result, Theorem 1 of Section 2 below, proves that weak – and if f is totally convex then strong – convergence of $\mathrm{Prox}^f_{\varphi_n}(\xi_n)$ to $\mathrm{Prox}^f_{\varphi}(\xi)$ as M-$\lim_{n\to\infty}\varphi_n = \varphi$ and $\lim_{n\to\infty}\xi_n = \xi$ can be ensured when $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above for Legendre functions f which are coercive (i.e., $\lim_{\|x\|\to\infty} f(x) = \infty$) and have the property that $\{f + \varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f + \varphi$. Note that the requirement that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above is equivalent to the condition that there exists a real number q such that for all $x \in X$

$$\varphi_n(x) \geq \langle \xi_n, x\rangle - q,\ \forall n \in \mathbb{N}. \tag{1.16}$$

This requirement does not imply uniform boundedness from below of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ unless $\xi_n = 0^*$ for all $n \in \mathbb{N}$. However, if $\{\varphi_n^*(0^*)\}_{n\in\mathbb{N}}$ is bounded from above (i.e., if $\{\varphi_n\}_{n\in\mathbb{N}}$ is uniformly bonded from below), then the main result in [**19**] guarantees the conclusion of Theorem 1 in our current paper without the additional requirement that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ should be bounded from above, but provided that f is better conditioned than we require here.

1.4 An open problem. In view of Theorem 3.66 in [**6**] which shows that Mosco convergence of the sequence of proper, lower semicontinuous convex functions $\{\varphi_n\}_{n\in\mathbb{N}}$ to the proper, lower semicontinuous convex function φ implies graphical convergence (see [**6**, Definition 3.58]) of the sequence of operators $\{\partial\varphi_n\}_{n\in\mathbb{N}}$ to $\partial\varphi$, the problem of Mosco stability for the proximal mapping can be seen as an instance of the following more general problem: Given a sequence of maximal monotone operators $A_n : X \to 2^{X^*}$, $n \in \mathbb{N}$, which converges graphically to some maximal monotone operator $A : X \to 2^{X^*}$, does the sequence of protoresolvents $(\nabla f + A_n)^{-1}$ converge in a stable manner to $(\nabla f + A)^{-1}$? In other words, the question is whether the weak/strong limit of $(\nabla f + A_n)^{-1}(\xi_n)$ is exactly $(\nabla f + A)^{-1}(\xi)$ when $\{\xi_n\}_{n\in\mathbb{N}}$ converges to ξ and A_n converges graphically to A. In the case of a Hilbert space X provided with the function $f = \frac{1}{2}\|\cdot\|^2$, strong pointwise convergence of $(\nabla f + A_n)^{-1}$ to $(\nabla f + A)^{-1}$ results from [**6**, Theorem 3.60]. Does this also happen in not necessarily hilbertian Banach spaces X provided with a totally convex Legendre function f? Theorem 1 proved in this paper, as well as the main result in [**19**], give sufficient conditions in this sense for the case of operators A_n

and A which are maximal cyclically monotone (i.e., subgradients of lower semicontinuous convex functions). Whether it is possible to extrapolate those results to arbitrary maximal monotone operators A_n and A (which are not necessarily cyclically monotone), is an interesting question whose answer we do not know. An affirmative answer to this question could help analyze the convergence behavior under data perturbations of algorithms for determining zeros of monotone operators based on Eckstein [**21**] type of generalized resolvents whose convergence theories were developed along the last decade in [**3**], [**4**], [**9**], [**10**], [**11**], [**12**] (see also the references of these works).

2. A stability theorem for the proximal mapping

In this section we establish a set of sufficient conditions for Mosco stability of the proximal mapping $\mathrm{Prox}_{\varphi}^{f}$. Analyzing our Mosco stability theorem for $\mathrm{Prox}_{\varphi}^{f}$, given below, one should observe that conditions (A) and (B) are only needed for ensuring that $\{f+\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f+\varphi$ when M-$\lim_{n\to\infty}\varphi_n=\varphi$. Alternative conditions for this to happen can be derived from [**24**, Theorem 5] and [**30**, Theorem 30(h)] and they can be used as replacements of (A) and (B) (see also Corollary 2 in the next section).

Theorem 1. *Suppose that the Legendre function f is coercive and $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty}\varphi_n=\varphi$. If any of the following conditions is satisfied*

(A) The function f has open domain;

(B) The function $f\,|_{\mathrm{dom}\, f}$, the restriction of f to its domain, is continuous and $\mathrm{dom}\,\varphi_n\subseteq\mathrm{dom}\,f$, *$(n\in\mathbb{N})$;*

and if $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence contained in $\mathrm{int}\,\mathrm{dom}\,f^*$ *such that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above and $\xi:=\lim_{n\to\infty}\xi_n\in\mathrm{int}\,\mathrm{dom}\,f^*$, then*

$$\text{w-}\lim_{n\to\infty}\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)=\mathrm{Prox}_{\varphi}^{f}(\xi) \tag{2.1}$$

and

$$\lim_{n\to\infty}\mathrm{Env}_{\varphi_n}^{f}(\xi_n)=\mathrm{Env}_{\varphi}^{f}(\xi). \tag{2.2}$$

Moreover, if the function f is also totally convex, then the convergence in (2.1) *is strong, that is,*

$$\lim_{n\to\infty}\mathrm{Prox}_{\varphi_n}^{f}(\xi_n)=\mathrm{Prox}_{\varphi}^{f}(\xi). \tag{2.3}$$

Proof. Denote

$$\hat{x}=\mathrm{Prox}_{\varphi}^{f}(\xi)\ \text{and}\ \hat{x}_n=\mathrm{Prox}_{\varphi_n}^{f}(\xi_n). \tag{2.4}$$

By (1.13) we have that, for each $x\in X$,

$$\varphi_n(\hat{x}_n)+W^f(\xi_n,\hat{x}_n)\le\varphi_n(x)+W^f(\xi_n,x),\ \forall n\in\mathbb{N}. \tag{2.5}$$

Hence, by (1.9) we have

$$\langle\xi_n,\hat{x}_n\rangle-(\varphi_n(\hat{x}_n)+f(\hat{x}_n))\ge\langle\xi_n,x\rangle-(\varphi_n(x)+f(x))\,,\ \forall n\in\mathbb{N},$$

whenever $x\in X$. Taking the supremum upon $x\in X$ in this inequality we get

$$\begin{aligned}\langle\xi_n,\hat{x}_n\rangle-(\varphi_n(\hat{x}_n)+f(\hat{x}_n)) &\ge (\varphi_n+f)^*(\xi_n)\\ &\ge \langle\xi_n,\hat{x}_n\rangle-(\varphi_n(\hat{x}_n)+f(\hat{x}_n))\,,\end{aligned}$$

for each $n \in \mathbb{N}$ and this implies

$$\langle \xi_n, \hat{x}_n \rangle - (\varphi_n(\hat{x}_n) + f(\hat{x}_n)) = (\varphi_n + f)^* (\xi_n), \ \forall n \in \mathbb{N}. \tag{2.6}$$

Now we are going to establish the following fact which may be well-known but we do not have a specific reference for it:

Claim 1: *The sequence $\{f + \varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to $f + \varphi$.*

In order to prove this claim we verify conditions $(M1)$ and $(M2)$ given above. To this end, let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in X and let x be its weak limit. Then

$$\liminf_{n\to\infty} \left(f + \varphi_n\right)(x_n) \geq \liminf_{n\to\infty} f(x_n) + \liminf_{n\to\infty} \varphi_n(x_n) \geq f(x) + \varphi(x).$$

where the last inequality holds because f is convex and lower semicontinuous (and, hence, weakly lower semicontinuous) and because, by hypothesis, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to φ (and, hence, it satisfies $(M1)$). Consequently, the sequence $\{f + \varphi_n\}_{n\in\mathbb{N}}$ and the function $f + \varphi$ satisfy $(M1)$. Now, in order to verify $(M2)$, let $u \in X$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in X such that (1.14) holds (a sequence like that exists because $\text{M-lim}_{n\to\infty} \varphi_n = \varphi$). In view of the validity of $(M1)$, it is sufficient to prove that

$$\limsup_{n\to\infty} \left(f(u_n) + \varphi_n(u_n)\right) \leq f(u) + \varphi(u). \tag{2.7}$$

We distinguish the following possible situations.

Case 1: If $u \notin \operatorname{dom} f$, then

$$\begin{aligned} \limsup_{n\to\infty} \left(f(u_n) + \varphi_n(u_n)\right) &\leq \limsup_{n\to\infty} f(u_n) + \limsup_{n\to\infty} \varphi_n(u_n) \\ &\leq \infty = f(u) + \varphi(u), \end{aligned} \tag{2.8}$$

that is, (2.7) holds.

Case 2: Suppose that $u \in \operatorname{dom} f$. If $u \in \operatorname{int} \operatorname{dom} f$, then there exists a positive integer n_0 such that $u_n \in \operatorname{int} \operatorname{dom} f$ for all $n \geq n_0$. Taking into account that, being lower semicontinuous, f is continuous on $\operatorname{int} \operatorname{dom} f$, this implies

$$\begin{aligned} \limsup_{n\to\infty} (f + \varphi_n)(u_n) &\leq \limsup_{n\to\infty} f(u_n) + \limsup_{n\to\infty} \varphi_n(u_n) \\ &= \lim_{n\to\infty} f(u_n) + \lim_{n\to\infty} \varphi_n(u_n) = f(u) + \varphi(u), \end{aligned} \tag{2.9}$$

showing that $(M2)$ holds in this situation. Hence, if condition (A) is satisfied, then (2.7) is true in all possible cases. Also, if (B) is satisfied, then (2.7) is true whenever u is not an element of the boundary of $\operatorname{dom} f$. Now, assume that condition (B) is satisfied and u is a boundary point of $\operatorname{dom} f$. In this situation, if there are infinitely many vectors u_n such that $u_n \notin \operatorname{dom} f$, then

$$\limsup_{n\to\infty} f(u_n) = \infty = \limsup_{n\to\infty} \varphi_n(u_n),$$

because, by (B), if $u_n \notin \operatorname{dom} f$, then $u_n \notin \operatorname{dom} \varphi_n$. Hence, according to (1.14), we deduce that $\varphi(u) = \limsup_{n\to\infty} \varphi_n(u_n) = \infty$ and, thus, (2.8) is true and, by consequence, (2.7) is also true. If all but finitely many vectors u_n are contained in $\operatorname{dom} f$, then

$$\limsup_{n\to\infty} f(u_n) = \lim_{n\to\infty} f(u_n) = f(u),$$

because of the continuity of $f\mid_{\operatorname{dom} f}$. By (1.14) this implies (2.9) and, thus, (2.7) is true in this situation too. Hence, when (B) holds, condition $(M2)$ is satisfied in all possible situations. This proves Claim 1.

Now we are going to establish the following fact:

Claim 2: *The sequence* $\{\hat{x}_n\}_{n\in\mathbb{N}}$ *defined by* (2.4) *is bounded.*

In order to prove this claim, suppose by contradiction that the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is not bounded. Then there exists a subsequence $\{\hat{x}_{k_n}\}_{n\in\mathbb{N}}$ of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|\hat{x}_{k_n}\| = +\infty$. Since, by hypothesis, the function f is coercive, we deduce that

$$\lim_{n\to\infty} f(\hat{x}_{k_n}) = +\infty. \tag{2.10}$$

According to (2.6), we have

$$f(\hat{x}_{k_n}) + \left(\varphi_{k_n}+f\right)^*(\xi_{k_n}) = \langle \xi, \hat{x}_{k_n}\rangle - \varphi_{k_n}(\hat{x}_{k_n}) \le \varphi^*_{k_n}(\xi_{k_n}),\ \forall n\in\mathbb{N}. \tag{2.11}$$

Theorem 3.18 in [**6**, p. 295] guarantees that, if ψ and ψ_n, $n\in\mathbb{N}$, are proper, lower semicontinuous convex functions on X and M-$\lim_{n\to\infty}\psi_n = \psi$, then we have M-$\lim_{n\to\infty}\psi_n^* = \psi^*$. This fact, combined with Claim 1 which shows that M-$\lim_{n\to\infty}(\varphi_n+f) = \varphi+f$, implies that

$$M-\lim_{n\to\infty}(\varphi_n+f)^* = (\varphi+f)^*.$$

Therefore,

$$M-\lim_{n\to\infty}\left(\varphi_{k_n}+f\right)^* = (\varphi+f)^*.$$

This implies (using $(M1)$ applied to the convergent sequence $\left\{\xi_{k_n}\right\}_{n\in\mathbb{N}}$ in X^*) that

$$\liminf_{n\to\infty}\left(\varphi_{k_n}+f\right)^*(\xi_{k_n}) \ge (\varphi+f)^*(\xi). \tag{2.12}$$

By (2.4) and (1.8) we have that

$$\varphi(\hat{x}) + W^f(\xi,\hat{x}) \le \varphi(x) + W^f(\xi,x),\ \forall x\in X. \tag{2.13}$$

By the definition of the Fenchel conjugate, (2.13) and (1.9) one deduces that

$$\begin{aligned}(\varphi+f)^*(\xi) &\ge \langle\xi,\hat{x}\rangle - (\varphi(\hat{x}) + f(\hat{x})) \\ &\ge \langle\xi,x\rangle - (\varphi(x)+f(x)),\ \forall x\in X.\end{aligned}$$

Taking the supremum upon $x\in X$ in this inequality we deduce

$$(\varphi+f)^*(\xi) = \langle\xi,\hat{x}\rangle - (\varphi(\hat{x}) + f(\hat{x})). \tag{2.14}$$

By (1.11) we have that

$$\hat{x}\in \operatorname{dom}\partial\varphi\cap\operatorname{dom} f\subseteq \operatorname{dom}\varphi\cap\operatorname{dom} f,$$

showing that $\varphi(\hat{x}) + f(\hat{x})$ is finite. Hence, by (2.14), $(\varphi+f)^*(\xi)$ is finite too. Thus, by (2.12),

$$\liminf_{n\to\infty}\left(\varphi_{k_n}+f\right)^*(\xi_{k_n}) > -\infty. \tag{2.15}$$

Taking lim inf as $n\to\infty$ on both sides of (2.11) we deduce that

$$\lim_{n\to\infty} f(\hat{x}_{k_n}) + \liminf_{n\to\infty}\left(\varphi_{k_n}+f\right)^*(\xi_{k_n}) \le \liminf_{n\to\infty}\varphi^*_{k_n}(\xi_{k_n}).$$

This, (2.10) and (2.15) imply that $\liminf\limits_{n\to\infty}\varphi^*_{k_n}(\xi_{k_n}) = +\infty$, that is, $\lim_{n\to\infty}\varphi^*_{k_n}(\xi_{k_n}) = +\infty$, which contradicts the boundedness of $\{\varphi^*_n(\xi_n)\}_{n\in\mathbb{N}}$. So, the proof of Claim 2 is complete.

The sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ being bounded in the reflexive space X, has weak cluster points. The claim we prove below shows that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is weakly convergent to $\hat{x}$ and, consequently, formula (2.1) holds.

Claim 3: *The only weak cluster point of* $\{\hat{x}_n\}_{n\in\mathbb{N}}$ *is* $\hat{x}$.

In order to prove Claim 3 let v be a weak cluster point of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ and let $\{\hat{x}_{i_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ such that $\text{w-lim}_{n\to\infty}\hat{x}_{i_n} = v$. Let u be any vector in $\operatorname{dom} f \cap \operatorname{dom}\varphi$. Since $\text{M-lim}_{n\to\infty}\varphi_n = \varphi$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty}\varphi_n(u_n) = \varphi(u). \tag{2.16}$$

The function f being convex and lower semicontinuous is also weakly lower semicontinuous. The sequences $\left\{f^*(\xi_{i_n})\right\}_{n\in\mathbb{N}}$ and $\left\{\left\langle \xi_{i_n}, \hat{x}_{i_n}\right\rangle\right\}_{n\in\mathbb{N}}$ converge to $f^*(\xi)$ and $\langle \xi, v\rangle$, respectively. Consequently, we have

$$\liminf_{n\to\infty} W^f(\xi_{i_n}, \hat{x}_{i_n}) \geq \liminf_{n\to\infty} f(\hat{x}_{i_n}) + \liminf_{n\to\infty}\left[f^*(\xi_{i_n}) - \left\langle \xi_{i_n}, \hat{x}_{i_n}\right\rangle\right] \tag{2.17}$$

$$\geq f(v) - \langle \xi, v\rangle + f^*(\xi) = W^f(\xi, v).$$

Due to the Mosco convergence of $\{\varphi_n\}_{n\in\mathbb{N}}$ (and, hence, of $\left\{\varphi_{i_n}\right\}_{n\in\mathbb{N}}$) to φ, to (2.17), and to (2.5) we deduce that

$$\varphi(v) + W^f(\xi, v) \leq \tag{2.18}$$

$$\liminf_{n\to\infty}\varphi_{i_n}(\hat{x}_{i_n}) + \liminf_{n\to\infty} W^f(\xi_{i_n}, \hat{x}_{i_n})$$

$$\leq \liminf_{n\to\infty}\operatorname{Env}^f_{\varphi_{i_n}}(\xi_{i_n}) \leq \limsup_{n\to\infty}\operatorname{Env}^f_{\varphi_{i_n}}(\xi_{i_n})$$

$$\leq \limsup_{n\to\infty}\left\{\varphi_{i_n}(u_{i_n}) + W^f(\xi_{i_n}, u_{i_n})\right\} = \varphi(u) + W^f(\xi, u).$$

Since u was arbitrarily chosen in $\operatorname{dom} f \cap \operatorname{dom}\varphi$, it follows that $v = \hat{x}$ and this proves Claim 3.

Now we are in position to show that (2.2) is also true. If we prove that, then the strong convergence of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ to $\hat{x}$ (i.e., (2.3)) when f is also totally convex results from [**19**, Theorem 2.1] and the proof of our theorem is complete. For proving (2.2) observe that, according to (1.11), the vector $\hat{x}$ belongs to $\operatorname{int}\operatorname{dom} f \cap \operatorname{dom}\varphi$ and, therefore, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that (2.16) holds for $u = \hat{x}$. Since the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $v = \hat{x}$, the inequalities and equality in (2.18) remain true when v is replaced by $\hat{x}$ and i_n is replaced by n. Therefore, taking into account (1.13) and (2.4), we deduce

$$\operatorname{Env}^f_{\varphi}(\xi) = \varphi(\hat{x}) + W^f(\xi, \hat{x})$$

$$\leq \liminf_{n\to\infty}\operatorname{Env}^f_{\varphi_n}(\xi_n) \leq \limsup_{n\to\infty}\operatorname{Env}^f_{\varphi_n}(\xi_n)$$

$$= \varphi(\hat{x}) + W^f(\xi, \hat{x})$$

and this implies (2.2). □

3. Consequences of the stability theorem

The following result shows that Theorem 1 applies to any constant sequence $\xi_n = \xi \in \operatorname{ran} \partial\varphi \cap \operatorname{int} \operatorname{dom} f^*$ since, for any such vector ξ, the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above.

Corollary 1. *Suppose that the Legendre function f is coercive and $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty} \varphi_n = \varphi$. If any of the conditions (A) or (B) of Theorem 1 is satisfied, and if $\xi \in \operatorname{ran} \partial\varphi \cap \operatorname{int} \operatorname{dom} f^*$, then*

$$\text{w-}\lim_{n\to\infty} \operatorname{Prox}_{\varphi_n}^f(\xi) = \operatorname{Prox}_{\varphi}^f(\xi) \tag{3.1}$$

and

$$\lim_{n\to\infty} \operatorname{Env}_{\varphi_n}^f(\xi) = \operatorname{Env}_{\varphi}^f(\xi). \tag{3.2}$$

Moreover, if the function f is also totally convex, then the convergence in (3.1) *is strong.*

Proof. According to Theorem 1, it is sufficient to show that if $\xi \in \operatorname{ran} \partial\varphi$, then the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above. To this end, let $\bar{x} \in X$ be such that $\xi \in \partial\varphi(\bar{x})$. Then, by the convexity of φ, we have

$$\varphi(x) - \varphi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall x \in X,$$

showing that

$$\varphi(x) \geq \langle \xi, x \rangle - q, \ \forall x \in X, \tag{3.3}$$

where $q = \langle \xi, \bar{x} \rangle - \varphi(\bar{x})$ is a real number because $\bar{x} \in \operatorname{dom} \partial\varphi \subseteq \operatorname{dom} \varphi$. By the hypothesis that M-$\lim_{n\to\infty} \varphi_n = \varphi$ combined with (3.3) we deduce (see $(M1)$) that for any $x \in X$

$$q \geq \langle \xi, x \rangle - \varphi(x) \geq \langle \xi, x \rangle - \liminf_{n\to\infty} \varphi_n(x) = \limsup_{n\to\infty} \left[\langle \xi, x \rangle - \varphi_n(x) \right].$$

Hence,

$$q \geq \sup_{x\in X} \limsup_{n\to\infty} \left[\langle \xi, x \rangle - \varphi_n(x) \right] = \limsup_{n\to\infty} \sup_{x\in X} \left[\langle \xi, x \rangle - \varphi_n(x) \right] = \limsup_{n\to\infty} \varphi_n^*(\xi),$$

showing that the sequence $\{\varphi_n^*(\xi)\}_{n\in\mathbb{N}}$ is bounded from above. □

It is meaningful to note that, if the Banach space X has finite dimension, then conditions (A) and (B) involved in Theorem 1 can be replaced by the requirement that

$$\operatorname{int} \operatorname{dom} \varphi \neq \varnothing. \tag{3.4}$$

To see that, note that conditions (A) and (B) are only used in the proof of Theorem 1 in order to ensure validity of Claim 1. Clearly, if (3.4) holds, then we also have that $0 \in \operatorname{int} (\operatorname{dom} \varphi - \operatorname{dom} f)$. Now, according to [**24**, Theorem 5], if $\dim X < \infty$, if $0 \in \operatorname{int} (\operatorname{dom} \varphi - \operatorname{dom} f)$ and if M-$\lim_{n\to\infty} \varphi_n = \varphi$, then M-$\lim_{n\to\infty}(f + \varphi_n) = f + \varphi$, that is, Claim 1 is satisfied. Hence, we deduce the following result:

Corollary 2. *Suppose that* $\dim X < \infty$ *and that the Legendre function f is coercive. If $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ are functions contained in $\mathcal{F}_f$ such that M-$\lim_{n\to\infty} \varphi_n = \varphi$ and* (3.4) *holds, and if $\{\xi_n\}_{n\in\mathbb{N}}$ is a convergent sequence contained in* $\operatorname{int} \operatorname{dom} f^*$ *such that $\{\varphi_n^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above and $\xi := \lim_{n\to\infty} \xi_n \in \operatorname{int} \operatorname{dom} f^*$, then* (2.1) *and* (2.2) *are true.*

Combining [**33**, Theorem 1 and Proposition 1] with (1.6), (1.10) and (1.11) one can see that the operator $\mathrm{Prox}_{\varphi}^{f}(\cdot)$ with $\varphi \in \mathcal{F}_f$ is maximal monotone and norm to weak continuous on $\operatorname{int}\operatorname{dom} f^*$. In other words, even if the function f is not coercive and even if none of the conditions (A) and (B) is satisfied, the equality (2.1) holds for any constant sequence $\varphi_n = \varphi \in \mathcal{F}_f$ and for any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ which is contained and converges in $\operatorname{int}\operatorname{dom} f^*$. A careful analysis of the proof of Theorem 1 shows that we have already proved that (2.1) implies (2.2). Also, carefully analyzing the proof of Theorem 1 one can observe that, if $\varphi_n = \varphi \in \mathcal{F}_f$ for all $n \in \mathbb{N}$, then the conditions (A) and (B) are superfluous (because in this case the conclusion of Claim 1 remains true even if these conditions do not hold). These remarks lead us to the following result:

Corollary 3. *If the Legendre function f is coercive and totally convex and if $\varphi \in \mathcal{F}_f$, then the following statements are true:*

(i) If B is a nonempty and bounded subset $\operatorname{dom}\partial\varphi$, then $\mathrm{Prox}_{\varphi}^{f}(\cdot)$ is norm to norm continuous on $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$;

(ii) If φ^* is bounded from above on bounded subsets of $\operatorname{int}\operatorname{dom} f^* \cap \operatorname{ran}\partial\varphi$, then $\mathrm{Prox}_{\varphi}^{f}(\cdot)$ is norm to norm continuous on $\operatorname{ran}\partial\varphi \cap \operatorname{int}\operatorname{dom} f^*$.

Proof. (i) Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ and ξ are contained in $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$ and satisfy $\lim_{n\to\infty} \xi_n = \xi$. Then, for each $n \in \mathbb{N}$, there exists a vector $\bar{x}_n \in B$ such that $\xi_n \in \partial\varphi(\bar{x}_n)$. By the convexity of φ we deduce that for any $x \in X$

$$\varphi(x) - \varphi(\bar{x}_n) \geq \langle \xi_n, x - \bar{x}_n \rangle, \ \forall n \in \mathbb{N}.$$

Hence, for any $x \in X$

$$\begin{aligned} \varphi(x) &\geq \langle \xi_n, x \rangle - \langle \xi_n, \bar{x}_n \rangle + \varphi(\bar{x}_n) \\ &\geq \langle \xi_n, x \rangle - \|\xi_n\|_* \|\bar{x}_n\| + \varphi(\bar{x}_n), \end{aligned} \tag{3.5}$$

where the sequence $\{\|\xi_n\|_* \|\bar{x}_n\|\}_{n\in\mathbb{N}}$ is bounded because both sequences $\{\|\xi_n\|_*\}_{n\in\mathbb{N}}$ and $\{\|\bar{x}_n\|\}_{n\in\mathbb{N}}$ are bounded, and the sequence $\{\varphi(\bar{x}_n)\}_{n\in\mathbb{N}}$ is bounded from below because $\{\bar{x}_n\}_{k\in\mathbb{N}}$ is contained in $\operatorname{dom}\varphi \cap \operatorname{int}\operatorname{dom} f$ and (1.7) holds. These facts, combined with (3.5) show that there exists a real number q such that for any $x \in X$

$$\varphi(x) \geq \langle \xi_n, x \rangle - q, \ \forall n \in \mathbb{N}.$$

In other words, the constant sequence $\varphi_n = \varphi$ satisfies (1.16) and, thus, the sequence $\{\varphi^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above. Applying Theorem 1 to the constant sequence $\varphi_n = \varphi$ and taking into account the remarks preceding this Corollary, we deduce that (2.3) holds in this case, i.e., $\mathrm{Prox}_{\varphi}^{f}(\cdot)$ is norm to norm continuous on $\partial\varphi(B) \cap \operatorname{int}\operatorname{dom} f^*$.

(ii) Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ and ξ are contained in $\operatorname{int}\operatorname{dom} f^* \cap \operatorname{ran}\partial\varphi$ and satisfy $\lim_{n\to\infty} \xi_n = \xi$. Then the sequence $\{\varphi^*(\xi_n)\}_{n\in\mathbb{N}}$ is bounded from above because $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded as being convergent. Application of Theorem 1 shows that (2.3) holds in this case too. □

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