Iterative Methods for Solving Systems of Variational Inequalities in Reflexive Banach Spaces

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Abstract. We prove strong convergence theorems for three iterative algorithms which approximate solutions to systems of variational inequalities for mappings of monotone type. All the theorems are set in reflexive Banach spaces and take into account possible computational errors.

1. Introduction

Given a nonempty, closed and convex subset K of a Banach space X, and a mapping $A : X \to 2^{X^*}$, the corresponding variational inequality is defined as follows:

$$\text{find } \bar{x} \in K \text{ such that there exists } \xi \in A(\bar{x}) \text{ with } \langle \xi, y - \bar{x} \rangle \geq 0 \ \ \forall y \in K. \tag{1.1}$$

The solution set of (1.1) is denoted by $VI(K, A)$.

Variational inequalities have turned out to be very useful in studying optimization problems, differential equations, minimax theorems and in certain applications to mechanics and economic theory. Important practical situations motivate the study of systems of variational inequalities (see [**19**] and the references therein). For instance, the flow of fluid through a fissured porous medium and certain models of plasticity lead to such problems (see, for instance, [**38**]).

Because of their importance, variational inequalities have been extensively analyzed in the literature (see, for example, [**23, 30, 40**] and the references therein). Usually either the monotonicity or a generalized monotonicity property of the mapping A play a crucial role in these investigations.

The aim of this paper is to present several iterative methods for solving systems of variational inequalities for different types of monotone-like mappings. Our methods are inspired by [**17, 24, 34, 35**], where iterative algorithms for finding zeroes of set-valued mappings are constructed using Bregman distances corresponding to totally convex functions. In contrast with [**17**], where only weak convergence is established, in all our results here we show that our algorithms converge strongly.

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Key words and phrases. Banach space, Bregman distance, Bregman firmly nonexpansive operator, Bregman inverse strongly monotone mapping, Bregman projection, hemicontinuous mapping, iterative algorithm, Legendre function, monotone mapping, pseudomonotone mapping, totally convex function, variational inequality.

The paper is organized in the following way. In the next section we present the preliminaries that are needed in our work. This section is divided into three subsections. The first one (Subsection 2.1) is devoted to functions while the second (Subsection 2.2) concerns (set-valued) mappings of monotone type. In the last subsection (Subsection 2.3) we deal with certain classes of Bregman nonexpansive operators. In the next three sections (Sections 3, 4 and 5) we present several algorithms for solving systems of variational inequalities corresponding to Bregman inverse strongly monotone, pseudomonotone and hemicontinuous mappings, respectively. The main differences among these algorithms involve the monotonicity assumptions imposed on the mappings which govern the variational inequalities. In the last section we present several particular cases of our algorithms.

2. Preliminaries

All the results in this paper are set in a real reflexive Banach space X with dual space X^*. The norms in X and X^* are denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. The pairing $\langle \xi, x \rangle$ is defined by the action of $\xi \in X^*$ at $x \in X$, that is, $\langle \xi, x \rangle = \xi(x)$. The set of all real numbers is denoted by $\mathbb{R}$ while $\mathbb{N}$ denotes the set of nonnegative integers.

Let $f : X \to (-\infty, +\infty]$ be a function. The *domain* of f is defined to be

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

When $\operatorname{dom} f \neq \emptyset$ we say that f is *proper*. We denote by $\operatorname{int}\operatorname{dom} f$ the *interior* of the domain of f.

Throughout this paper, $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and convex function. The *Fenchel conjugate* of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x \rangle - f(x) : x \in X\}.$$

The aim of this section is to define and present the basic notions and facts that are needed in the sequel. We divide this section into three parts in the following way. The first one (Subsection 2.1) is devoted to functions while the second (Subsection 2.2) concerns (set-valued) mappings of monotone type. In the last part (Subsection 2.3) we deal with certain types of Bregman nonexpansive operators.

2.1. Facts about functions. Let $x \in \operatorname{int}\operatorname{dom} f$. For any $y \in X$, we define the *right-hand derivative* of f at x by

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

If the limit in (2.1) exists as $t \to 0$ for each y, then the function f is said to be *Gâteaux differentiable at x*. In this case, the *gradient* of f at x is the linear function $\nabla f(x)$ which is defined by $\langle \nabla f(x), y \rangle = f^\circ(x, y)$ for any $y \in X$ (see [**31**, Definition 1.3, p. 3]). The function f is called *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$.

When the limit in (2.1) is attained uniformly for any $y \in X$ with $\|y\| = 1$ we say that f is *Fréchet differentiable* at x. The function f is called *uniformly Fréchet differentiable on a bounded subset E* if the limit in (2.1) is attained uniformly for any $x \in E$ and for any $y \in X$ with $\|y\| = 1$. If this holds for any bounded subset of X, then f is said to be *uniformly Fréchet differentiable on bounded subsets* of X.

The following statement is essential for the proofs of our main results (*cf.* [**33**, Proposition 2.1, p. 474] and [**1**, Theorem 1.8, p. 13]).

PROPOSITION 1. *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the two assertions hold:*

(i) *f is uniformly continuous on bounded subsets of X;*
(ii) *∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Our main results hold for the following class of functions. The function f is called *Legendre* [**10**] if it satisfies the following two conditions:

(L1) f is Gâteaux differentiable and $\operatorname{int}\operatorname{dom} f \neq \emptyset$;
(L2) f^* is Gâteaux differentiable and $\operatorname{int}\operatorname{dom} f^* \neq \emptyset$.

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [**3**]. Their definition is equivalent to conditions (L1) and (L2) because X is assumed to be a reflexive Banach space (see [**3**, Theorems 5.4 and 5.6, p. 634]).

In reflexive spaces it is well-known that $\nabla f = (\nabla f^*)^{-1}$ (see [**8**, p. 83]). Combining this fact with conditions (L1) and (L2), we get

$$\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^* \text{ and } \operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f.$$

It also follows that f is Legendre if and only if f^* is Legendre (see [**3**, Corollary 5.5, p. 634]) and that the functions f and f^* are strictly convex on the interior of their respective domains.

When the Banach space X is smooth and strictly convex, in particular, a Hilbert space, the function $(1/p)\left\|\cdot\right\|^p$ with $p \in (1, \infty)$ is Legendre. For examples and more information regarding Legendre functions, see, for instance, [**2, 3**].

From now on we assume that the function $f : X \to (-\infty, +\infty]$ is also Legendre.

In order to obtain our main results in the context of general reflexive Banach spaces we will use the Bregman distance instead of the norm. The bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$, defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \tag{2.2}$$

is called the *Bregman distance with respect to* f (*cf.* [**11, 20**]). The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \tag{2.3}$$

The strong convergence results which we prove in this paper are based on the convexity of the function f. Since the strict convexity of f does not seem to guarantee strong convergence of our algorithms, we assume that f is totally convex. This assumption is stronger than strict convexity (see [**14**, Proposition 1.2.6(i), p. 27]), but less stringent than uniform convexity (see [**14**, Section 2.3, p. 92]).

According to [**14**, Section 1.2, p. 17] (see also [**13**]), the *modulus of total convexity at x of f* is the bifunction $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$\upsilon_f(x, t) := \inf\left\{D_f(y, x) : y \in \operatorname{dom} f,\ \|y - x\| = t\right\}.$$

The function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if $\upsilon_f(x, t) > 0$ whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int} \operatorname{dom} f$. Let E be a subset of X. We define the *modulus of total convexity of* f *on* E as follows:

$$\upsilon_f (E,t) := \inf \{\upsilon_f (x,t) : x \in E \cap \operatorname{int} \operatorname{dom} f\}, \quad t > 0.$$

If $\upsilon_f (E,t) > 0$ for any bounded subset E of X and for any $t > 0$, then we say that f is *totally convex on bounded subsets of* X. Examples of totally convex functions can be found, for instance, in [**9, 14, 18**].

We remark in passing that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [**18**, Theorem 2.10, p. 9]).

Recall that the function f is called *sequentially consistent* (see [**18**]) if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{int} \operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to\infty} D_f (y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The next two propositions turn out to be very useful in the proofs of our results. The second one follows from [**16**, Proposition 2.3, p. 39] and [**39**, Theorem 3.5.10, p. 164].

Proposition 2 (*cf.* [**14**, Lemma 2.1.2, p. 67]). *Let* $f : X \to (-\infty, +\infty]$ *be a Gâteaux differentiable function. Then* f *is totally convex on bounded subsets if and only if it is sequentially consistent.*

Proposition 3. *If* $f : X \to (-\infty, +\infty]$ *is Fréchet differentiable and totally convex, then* f *is cofinite, that is,* $\operatorname{dom} f^* = X^*$.

The next proposition exhibits an additional property of totally convex functions.

Proposition 4 (*cf.* [**34**, Lemma 3.1, p. 31]). *Suppose that the Gâteaux differentiable function* $f : X \to \mathbb{R}$ *is totally convex. Let* $x_0 \in X$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset X$. *If the sequence* $\{D_f (x_n, x_0)\}_{n\in\mathbb{N}}$ *is bounded, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded too.*

A function f is said to be *coercive* (respectively, *supercoercive*) [**4**] if $\lim_{\|x\|\to+\infty} f (x) = +\infty$ (respectively, $\lim_{\|x\|\to+\infty} (f (x) / \|x\|) = +\infty$).

The following result brings out the fact that the Bregman distance is nonsymmetric.

Proposition 5. *Let* $f : X \to \mathbb{R}$ *be a Legendre function such that* $\operatorname{dom} \nabla f^* = X^*$ *and* ∇f^* *is bounded on bounded subsets of* X^*. *Let* $x_0 \in X$ *and* $\{x_n\}_{n\in\mathbb{N}} \subset X$. *If* $\{D_f (x_0, x_n)\}_{n\in\mathbb{N}}$ *is bounded, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded too.*

Proof. According to [**3**, Theorem 3.3, p. 624], f is supercoercive because $\operatorname{dom} \nabla f^* = X^*$ and ∇f^* is bounded on bounded subsets of X^*. From [**3**, Lemma 7.3(viii), p. 642] it follows that $D_f (x_0, \cdot)$ is coercive. If the sequence $\{x_n\}_{n\in\mathbb{N}}$ were unbounded, then there would exist a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with $\|x_{n_k}\| \to \infty$. This, since $D_f (x_0, \cdot)$ is coercive, implies that $D_f (x_0, x_{n_k}) \to \infty$, which is a contradiction. Thus $\{x_n\}_{n\in\mathbb{N}}$ is indeed bounded, as claimed. □

We define the *Bregman projection* (*cf.* [**11**]) of x onto the nonempty, closed and convex set $K \subset \operatorname{dom} f$ as the necessarily unique vector $\operatorname{proj}^f_K (x) \in K$ which satisfies (see [**5**])

$$D_f \left(\operatorname{proj}^f_K (x), x\right) = \inf \{D_f (y, x) : y \in K\}.$$

Similarly to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex functions has a variational characterization.

Proposition 6 (*cf.* [**18**, Corollary 4.4, p. 23]). *Suppose that the Gâteaux differentiable function $f : X \to (-\infty, +\infty]$ is totally convex. Let $x \in \operatorname{int}\operatorname{dom} f$ and let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following conditions are equivalent:*

(i) *The vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;*
(ii) *The vector $\hat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \, \forall y \in K;$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \; \forall y \in K.$$

The following result will be the key tool for proving strong convergence in our main results (see Lemma 4 in Section 3).

Proposition 7 (*cf.* [**34**, Lemma 3.2, p. 31]). *Suppose that the Gâteaux differentiable function $f : X \to \mathbb{R}$ is totally convex. Let $x_0 \in X$ and let K be a nonempty, closed and convex subset of X. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and that any weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to K. If $D_f(x_n, x_0) \leq D_f\left(\operatorname{proj}_K^f(x_0), x_0\right)$ for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_K^f(x_0)$.*

2.2. Facts about mappings of monotone type. Let $A : X \to 2^{X^*}$ be a mapping. Recall that the set $\operatorname{dom} A = \{x \in X : Ax \neq \emptyset\}$ is called the *domain* of the mapping A. We say that A is a *monotone* mapping if for any $x, y \in \operatorname{dom} A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \implies \langle \xi - \eta, x - y \rangle \geq 0. \tag{2.4}$$

A monotone mapping A is said to be *maximal* if the graph of A is not a proper subset of the graph of any other monotone mapping. The mapping A is said to be *demiclosed* at $x \in \operatorname{dom} A$ if for any sequence $\{(x_n, \xi_n)\}_{n\in\mathbb{N}}$ in $X \times X^*$ we have

$$\left.\begin{array}{c} x_n \rightharpoonup x \\ \xi_n \in Ax_n, \, n \in \mathbb{N} \\ \xi_n \to \xi \end{array}\right\} \implies \xi \in Ax. \tag{2.5}$$

If the mapping A is single-valued, then we write $A : \operatorname{dom} A \subset X \to X^*$, or $A : X \to X^*$, for short.

The mapping $A : X \to X^*$ is called *hemicontinuous* if for any $x \in \operatorname{dom} A$ we have

$$\left.\begin{array}{c} x + t_n y \in \operatorname{dom} A, \, y \in X \\ \lim_{n\to\infty} t_n = 0^+ \end{array}\right\} \implies A(x + t_n y) \rightharpoonup Ax. \tag{2.6}$$

Let $A : X \to 2^{X^*}$ be a mapping. The *resolvent* of A is the operator $\operatorname{Res}_A^f : X \to 2^X$ defined by

$$\operatorname{Res}_A^f = (\nabla f + A)^{-1} \circ \nabla f. \tag{2.7}$$

The following class of mappings was first introduced by Butnariu and Kassay in [**17**]. Assume that the mapping A satisfies the following range condition with respect to the Legendre function f:

$$\operatorname{ran}\left(\nabla f - A\right) \subset \operatorname{ran}\left(\nabla f\right). \tag{2.8}$$

Remark 1. *Observe that condition (2.8) is satisfied by many classes of functions and mappings. Suppose, for example, that f is cofinite, that is, $\operatorname{dom} f^* = X^*$. Note that if f is Fréchet differentiable and totally convex, then it is indeed cofinite (see Proposition 3). In our case, since f is also Legendre, we have $\operatorname{ran} \nabla f = \operatorname{int} \operatorname{dom} f^* = X^*$. Therefore condition (2.8) is always satisfied in our setting without any additional assumptions on the mapping A.*

Let Y be a subset of the space X. The mapping $A : X \to 2^{X^*}$ is called *Bregman inverse strongly monotone* (BISM for short) on the set Y if

$$Y \bigcap (\operatorname{dom} A) \bigcap (\operatorname{int} \operatorname{dom} f) \neq \emptyset, \tag{2.9}$$

and for any $x, y \in Y \bigcap (\operatorname{int} \operatorname{dom} f)$, and $\xi \in Ax$, $\eta \in Ay$, we have

$$\left\langle \xi - \eta, \nabla f^* \left(\nabla f \left(x\right) - \xi\right) - \nabla f^* \left(\nabla f \left(y\right) - \eta\right)\right\rangle \geq 0. \tag{2.10}$$

Remark 2. *The BISM class of mappings is a generalization of the class of firmly nonexpansive operators in Hilbert spaces. Indeed, if $f = (1/2) \left\|\cdot\right\|^2$, then $\nabla f = \nabla f^* = I$, where I is the identity operator, and (2.10) becomes*

$$\left\langle \xi - \eta, x - \xi - (y - \eta)\right\rangle \geq 0, \tag{2.11}$$

that is,

$$\left\|\xi - \eta\right\|^2 \leq \left\langle x - y, \xi - \eta\right\rangle. \tag{2.12}$$

In other words, A is a (single-valued) firmly nonexpansive operator.

The *anti-resolvent* $A^f : X \to 2^X$ of a mapping $A : X \to 2^{X^*}$ is defined by

$$A^f := \nabla f^* \circ \left(\nabla f - A\right). \tag{2.13}$$

Observe that $\operatorname{dom} A^f = (\operatorname{dom} A) \bigcap (\operatorname{int} \operatorname{dom} f)$ and $\operatorname{ran} A^f \subset \operatorname{int} \operatorname{dom} f$. For examples of BISM mappings and more information on this new class of mappings see [**17, 35**].

The following example shows that a BISM mapping might not be maximal monotone.

Example 1. *Let K be any closed, convex and proper subset of X. Let $A : X \to 2^{X^*}$ be any BISM mapping with $\operatorname{dom} A = K$ such that Ax is a bounded set for any $x \in X$. Then A is not maximal monotone. Indeed, $\operatorname{cl} K = K \neq X$, which means that $\operatorname{bdr} K = \operatorname{cl} K \setminus \operatorname{int} K \neq \emptyset$. Now for any $x \in \operatorname{bdr} K$ we know that Ax is a nonempty and bounded set. On the other hand, Ax is unbounded whenever A is maximal monotone, since we know that the image of a point on the boundary of the domain of a maximal monotone mapping, if non-empty, is unbounded because it contains a half-line.*

A very simple particular case is the following one: X is a Hilbert space, $f = (1/2) \left\|\cdot\right\|^2$ (in this case BISM reduces to firm nonexpansivity (see Remark 2)), K is a nonempty, closed, convex and bounded subset of X (e.g., a closed ball) and A is any single-valued BISM operator on K (e.g., the identity) and $\emptyset$ otherwise.

Problem 1. *Since a BISM mapping need not be maximal monotone, it is of interest to determine if it must be a monotone mapping.*

Recall that the mapping $A : X \to X^*$ is said to be *pseudomonotone* in the sense of Brezis (see [**12**]) if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ which converges weakly to $x \in \operatorname{dom} A$ and satisfies

$$\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle \le 0, \tag{2.14}$$

it follows that for each $y \in \operatorname{dom} A$,

$$\langle Ax, x - y\rangle \le \liminf_{n\to\infty} \langle Ax_n, x_n - y\rangle . \tag{2.15}$$

For more information on pseudomonotone mappings see, for instance, [**29, 40**] and the references therein.

The following result brings out the connection between hemicontinuous and pseudomonotone mappings.

Proposition 8 (*cf.* [**40**, Proposition 27.6(a), p. 586]). *If $A : X \to X^*$ is a monotone and hemicontinuous mapping, then A is pseudomonotone.*

2.3. Facts about Operators. Let K be a nonempty and convex subset of $\operatorname{int}\operatorname{dom} f$. An operator $T : K \to \operatorname{int}\operatorname{dom} f$ is called *Bregman firmly nonexpansive* (BFNE for short) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \tag{2.16}$$

for all $x, y \in K$. It is clear from the definition of the Bregman distance (2.2) that (2.16) is equivalent to

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \le D_f(Tx, y) + D_f(Ty, x) .$$

For more details on BFNE operators see [**4, 36**].

The *fixed point set* of an operator $T : K \to X$ is denoted by $F(T)$, that is, $F(T) := \{x \in K : x = Tx\}$.

Assume that $F(T) \ne \emptyset$. We say that $T : K \to \operatorname{int}\operatorname{dom} f$ is *quasi-Bregman firmly nonexpansive* (QBFNE) if for any $x \in K$ and $p \in F(T)$,

$$\langle \nabla f(x) - \nabla f(Tx), Tx - p\rangle \ge 0, \tag{2.17}$$

which is equivalent to

$$D_f(p, Tx) + D_f(Tx, x) \le D_f(p, x) . \tag{2.18}$$

It is clear that any quasi-Bregman firmly nonexpansive operator is *quasi-Bregman nonexpansive* (QBNE), that is, it satisfies

$$D_f(p, Tx) \le D_f(p, x) \tag{2.19}$$

for any $x \in K$ and for all $p \in F(T)$.

A point p in the closure of K is said to be an *asymptotic fixed point* of $T : K \to X$ (*cf.* [**32**]) if K contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The *asymptotic fixed point set* of T is denoted by $\widehat{F}(T)$.

Another type of Bregman nonexpansive operators was first introduced in [**21, 32**]. We say that an operator T is *Bregman strongly nonexpansive* (BSNE) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \le D_f(p, x) \tag{2.20}$$

for all $p \in \widehat{F}(T)$ and $x \in K$, and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n\to\infty} \left(D_f(p, x_n) - D_f(p, Tx_n)\right) = 0, \tag{2.21}$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0. \tag{2.22}$$

These operators have the following important property.

Proposition 9 (*cf.* [**32**, Lemmas 1 and 2, p. 314]). *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let K be a nonempty, closed and convex subset of X. Let $\{T_i : 1 \le i \le N\}$ be N BSNE operators from K into itself and put $T := T_N T_{N-1} \cdots T_1$. If the set*

$$\widehat{F} = \bigcap \left\{\widehat{F}(T_i) : 1 \le i \le N\right\}$$

is not empty, then $\widehat{F}(T) \subset \widehat{F}$. In addition, if $\widehat{F}(T)$ is nonempty, then T is BSNE with respect to $\widehat{F}(T)$.

In applications it seems that the assumption $\widehat{F}(T) = F(T)$ regarding the operator T is essential for the convergence of iterative methods. Therefore we recall the following result.

Proposition 10 (*cf.* [**36**, Lemma 15.6, p. 306]). *Let $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of X. Let K be a nonempty, closed and convex subset of X, and let $T : K \to X$ be a BFNE operator. Then $F(T) = \widehat{F}(T)$.*

The following remark shows that this condition holds for the composition of N BSNE operators when each operator satisfies it.

Remark 3. *Assume that $f : X \to \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of X. Let K be a nonempty, closed and convex subset of X. Let $\{T_i : 1 \le i \le N\}$ be N BSNE operators which satisfy $\widehat{F}(T_i) = F(T_i)$ for each $1 \le i \le N$ and let $T = T_N T_{N-1} \cdots T_1$. If*

$$\bigcap \{F(T_i) : 1 \le i \le N\}$$

and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \widehat{F}(T)$. Indeed, from Proposition 9 we get

$$F(T) \subset \widehat{F}(T) \subset \bigcap \left\{\widehat{F}(T_i) : 1 \le i \le N\right\} = \bigcap \{F(T_i) : 1 \le i \le N\} \subset F(T),$$

which implies that $F(T) = \widehat{F}(T)$, as claimed.

The following remark brings out the connections between the classes of operators defined above.

Remark 4. *Let $T : K \to \operatorname{int}\operatorname{dom} f$ be an operator such that $\widehat{F}(T) = F(T) \neq \emptyset$. It is easy to see that the following inclusions hold:*

$$BFNE \subset QBFNE \subset BSNE \subset QBNE.$$

From the definition of the anti-resolvent and [**17**, Lemma 3.5, p. 2109] we obtain the following proposition.

Proposition 11. *Let* $A : X \to 2^{X^*}$ *be a BISM mapping such that* $A^{-1}(0^*) \neq \emptyset$*. Let* $f : X \to \mathbb{R}$ *be a Legendre function which satisfies the range condition (2.8). Then the following statements hold:*

(i) $A^{-1}(0^*) = F\left(A^f\right)$*;*

(ii) *the anti-resolvent* A^f *is a BFNE operator. In addition,*

$$D_f\left(u, A^f x\right) + D_f\left(A^f x, x\right) \leq D_f(u, x)$$

for any $u \in A^{-1}(0^*)$ *and for all* $x \in \operatorname{dom} A^f$*.*

Let K be a nonempty, closed and convex subset of X and let $A : X \to X^*$ be a mapping. The variational inequality corresponding to such a mapping A is

$$\text{find } \bar{x} \in K \text{ such that } \langle A(\bar{x}), y - \bar{x}\rangle \geq 0 \ \ \forall y \in K. \tag{2.23}$$

The solution set of (2.23) is denoted by $VI(K, A)$.

In the following result we bring out the connections between the fixed point set of $\operatorname{proj}_K^f \circ A^f$ and the solution set of the variational inequality corresponding to a single-valued mapping.

Proposition 12. *Let* $A : X \to X^*$ *be a mapping. Let* $f : X \to (-\infty, +\infty]$ *be a Legendre and totally convex function which satisfies the range condition (2.8). If* K *is a nonempty, closed and convex subset of* X*, then* $VI(K, A) = F\left(\operatorname{proj}_K^f \circ A^f\right)$*.*

Proof. From Proposition 6(ii) we obtain that $x = \operatorname{proj}_K^f\left(A^f x\right)$ if and only if

$$\left\langle \nabla f\left(A^f x\right) - \nabla f(x), x - y\right\rangle \geq 0$$

for all $y \in K$. This is equivalent to

$$\langle (\nabla f - A)x - \nabla f(x), x - y\rangle \geq 0$$

for any $y \in K$, that is,

$$\langle -Ax, x - y\rangle \geq 0$$

for each $y \in K$, which is obviously equivalent to $x \in VI(K, A)$, as claimed. □

It is obvious that any zero of a mapping A which belongs to K is a solution of the variational inequality corresponding to A on the set K, that is, $A^{-1}(0^*) \cap K \subset VI(K, A)$. In the following result we show that the converse implication holds for single-valued BISM mappings.

Proposition 13. *Let* $f : X \to (-\infty, +\infty]$ *be a Legendre and totally convex function which satisfies the range condition (2.8). Let* K *be a nonempty, closed and convex subset of* $(\operatorname{dom} A) \bigcap (\operatorname{int}\operatorname{dom} f)$*. If the BISM mapping* $A : X \to X^*$ *satisfies* $Z := A^{-1}(0^*) \cap K \neq \emptyset$*, then* $VI(K, A) = Z$*.*

Proof. Let $x \in VI(K, A)$. By Proposition 12 we know that $x = \operatorname{proj}_K^f\left(A^f x\right)$. From Proposition 6(iiit) we now obtain that

$$D_f\left(u, \operatorname{proj}_K^f\left(A^f x\right)\right) + D_f\left(\operatorname{proj}_K^f\left(A^f x\right), A^f x\right) \leq D_f\left(u, A^f x\right)$$

for any $u \in K$. Hence from Proposition 11(ii) we get

$$\begin{aligned} D_f(u, x) + D_f\left(x, A^f x\right) &= D_f\left(u, \operatorname{proj}_K^f\left(A^f x\right)\right) + D_f\left(\operatorname{proj}_K^f\left(A^f x\right), A^f x\right) \\ &\leq D_f\left(u, A^f x\right) \leq D_f(u, x) \end{aligned}$$

for any $u \in Z$. This implies that $D_f\left(x, A^f x\right) = 0$. It now follows from [**3**, Lemma 7.3(vi), p. 642] that $x = A^f x$, that is, $x \in F\left(A^f\right)$, and from Proposition 11(i) we get that $x \in A^{-1}\left(0^*\right)$. Since $x = \operatorname{proj}_K^f\left(A^f x\right)$, it is clear that $x \in K$ and therefore $x \in Z$. Conversely, let $x \in Z$. Then $x \in K$ and $Ax = 0^*$, so it is obvious that (2.23) is satisfied. In other words, $x \in VI\left(K, A\right)$.

This completes the proof of Proposition 13. □

The following example shows that the assumption $Z \neq \emptyset$ in Proposition 13 is essential.

Example 2. *Let $X = \mathbb{R}$, $f = (1/2)\left\|\cdot\right\|^2$, $K = [1, +\infty)$ and let $A : \mathbb{R} \to \mathbb{R}$ be given by $Ax = x$ (the identity operator). This is obviously a BISM mapping (which in our case means that it is firmly nonexpansive (see Remark 2)) and all the assumptions of Proposition 13 hold, except $Z \neq \emptyset$. Indeed, we have $A^{-1}(0) = \{0\}$ and $0 \notin K$. However, $V = \{1\}$ since the only solution of the variational inequality $x\left(y - x\right) \geq 0$ for all $y \geq 1$ is $x = 1$ and therefore $Z = \emptyset$ is a* **proper** *subset of V.*

Bauschke, Borwein and Combettes [**4**] proved that when the mapping A is maximal monotone, then its resolvent $\operatorname{Res}_A^f(x)$ is a BFNE single-valued operator with full domain and we have

$$F\left(\operatorname{Res}_A^f(x)\right) = A^{-1}\left(0^*\right) \bigcap \left(\operatorname{int}\operatorname{dom} f\right).$$

3. Solving Variational Inequalities for BISM Mappings

In this section we present two algorithms for solving systems of variational inequalities corresponding to finitely many BISM mappings $\{A_i\}_{i=1}^N$. More precisely, let $\varepsilon > 0$ and let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^N K_i$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N BISM mappings such that $B\left(K_i, \varepsilon\right) \subset \operatorname{dom} A_i$ and $V := \bigcap_{i=1}^N VI\left(K_i, A_i\right) \neq \emptyset$, where $B\left(K_i, \varepsilon\right) := \{x \in X : d\left(x, K\right) < \varepsilon\}$ and $d\left(x, K\right) := \inf\left\{\|x - y\| : y \in K\right\}$. We consider the following two algorithms:

$$\begin{cases} x_0 \in K = \bigcap_{i=1}^N K_i, \\ y_n^i = A_i^f\left(x_n + e_n^i\right), \\ C_n^i = \left\{z \in K_i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in K : \left\langle \nabla f\left(x_0\right) - \nabla f\left(x_n\right), z - x_n\right\rangle \leq 0\right\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f\left(x_0\right), \ n = 0, 1, 2, \ldots, \end{cases} \tag{3.1}$$

and

$$\begin{cases} x_0 \in K = \bigcap_{i=1}^N K_i, \\ y_n^i = \operatorname{proj}_{K_i}^f\left(A_i^f\left(x_n + e_n^i\right)\right), \\ C_n^i = \left\{z \in K_i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in K : \left\langle \nabla f\left(x_0\right) - \nabla f\left(x_n\right), z - x_n\right\rangle \leq 0\right\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f\left(x_0\right), \ n = 0, 1, 2, \ldots, \end{cases} \tag{3.2}$$

where each $\left\{e_n^i\right\}_{n\in\mathbb{N}}$, $i=1,2,\ldots,N$, is a sequence of errors which satisfies $\left\|e_n^i\right\|<\varepsilon$ and $\lim_{n\to\infty} e_n^i=0$.

Since the proofs that these two algorithms generate sequences which converge strongly to a solution of the given system of variational inequalities are somewhat similar, we first prove several lemmata which are common to both proofs (and also to the proofs in Sections 4 and 5) and then present the statements and the proofs of our main results.

In order to prove our lemmata, we consider a more general version of these two algorithms. More precisely, we consider the following algorithm:

$$
\left\{\begin{array}{l}
x_0 \in K=\bigcap_{i=1}^N K_i, \\
y_n^i=T_n^i\left(x_n+e_n^i\right), \\
C_n^i=\left\{z \in K_i: D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n+e_n^i\right)\right\}, \\
C_n:=\bigcap_{i=1}^N C_n^i, \\
Q_n=\left\{z \in K:\left\langle\nabla f\left(x_0\right)-\nabla f\left(x_n\right), z-x_n\right\rangle \leq 0\right\}, \\
x_{n+1}=\operatorname{proj}_{C_n \cap Q_n}^f\left(x_0\right),\ n=0,1,2, \ldots,
\end{array}\right. \tag{3.3}
$$

where $T_n^i: \operatorname{dom} T_n^i \subset X \to X$ are given operators for each $i=1,2,\ldots,N$ and $n\in\mathbb{N}$. All our lemmata are proved under several assumptions, which we summarize as follows:

Condition 1. *Let $\varepsilon>0$ and let K_i, $i=1,2,\ldots,N$, be N nonempty, closed and convex subsets of X such that $K:=\bigcap_{i=1}^N K_i$. Let $T_n^i:\operatorname{dom}T_n^i\subset X\to X$, $i=1,2,\ldots,N$ and $n\in\mathbb{N}$, be QBNE operators such that $B\left(K_i,\varepsilon\right)\subset\operatorname{dom}T_n^i$ and $F:=\bigcap_{n\in\mathbb{N}}\bigcap_{i=1}^N F\left(T_n^i\right)\bigcap K\neq\emptyset$. Let $f:X\to\mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. Assume that, for each $i=1,2,\ldots,N$, the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}}\subset X$ satisfies $\left\|e_n^i\right\|<\varepsilon$ and $\lim_{n\to\infty}e_n^i=0$.*

Now we prove a sequence of lemmata.

Lemma 1. *Algorithm (3.3) is well defined.*

Proof. The point y_n^i is well defined for each $i=1,2,\ldots,N$ and $n\in\mathbb{N}$ because $B\left(K_i,\varepsilon\right)\subset\operatorname{dom}T_n^i$ and $\left\|e_n^i\right\|<\varepsilon$. Hence we only have to show that $\left\{x_n\right\}_{n\in\mathbb{N}}$ is well defined. To this end, we will prove that the Bregman projection onto $C_n\bigcap Q_n$ is well defined, that is, we need to show that $C_n\bigcap Q_n$ is a nonempty, closed and convex subset of X for each $n\in\mathbb{N}$. Since $x_0\in K$ and $Q_n\subset K$, this will also show that $x_n\in K$. Let $n\in\mathbb{N}$. It is not difficult to check that C_n^i are closed half-spaces for any $i=1,2,\ldots,N$. Hence their intersection C_n is a closed polyhedral set. It is also obvious that Q_n is a closed half-space. Let $u\in F$. For any $n\in\mathbb{N}$, we obtain from (2.19) that

$$
D_f\left(u, y_n^i\right)=D_f\left(u, T_n^i\left(x_n+e_n^i\right)\right) \leq D_f\left(u, x_n+e_n^i\right),
$$

which implies that $u\in C_n^i$. Since this holds for any $i=1,2,\ldots,N$, it follows that $u\in C_n$. Thus $F\subset C_n$ for any $n\in\mathbb{N}$. On the other hand, it is obvious that $F\subset Q_0=K$. Thus $F\subset C_0\bigcap Q_0$, and therefore $x_1=\operatorname{proj}_{C_0\cap Q_0}^f\left(x_0\right)$ is well defined. Now suppose that $F\subset C_{n-1}\bigcap Q_{n-1}$ for some $n\geq 1$. Then $x_n=\operatorname{proj}_{C_{n-1}\cap Q_{n-1}}^f\left(x_0\right)$ is well defined because $C_{n-1}\bigcap Q_{n-1}$ is a nonempty, closed and convex subset of X. So from Proposition 6(ii) we have

$$\langle \nabla f(x_0) - \nabla f(x_n), y - x_n \rangle \le 0$$

for any $y \in C_{n-1} \bigcap Q_{n-1}$. Hence we obtain that $F \subset Q_n$. Therefore $F \subset C_n \bigcap Q_n$ and so $C_n \bigcap Q_n$ is nonempty. Hence $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0)$ is well defined. Consequently, we see that $F \subset C_n \bigcap Q_n$ for any $n \in \mathbb{N}$. Thus the sequence we constructed is indeed well defined and satisfies (3.3), as claimed. □

From now on we fix an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ which is generated by Algorithm (3.3).

LEMMA 2. *The sequences* $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$, $\{x_n\}_{n \in \mathbb{N}}$ *and* $\{y_n^i\}_{n \in \mathbb{N}}$, $i = 1, 2, \ldots, N$, *are bounded.*

PROOF. It follows from the definition of Q_n and Proposition 6(ii) that $\mathrm{proj}^f_{Q_n}(x_0) = x_n$. Furthermore, by Proposition 6(iii), for each $u \in F$, we have

$$\begin{aligned} D_f(x_n, x_0) &= D_f\left(\mathrm{proj}^f_{Q_n}(x_0), x_0\right) \\ &\le D_f(u, x_0) - D_f\left(u, \mathrm{proj}^f_{Q_n}(x_0)\right) \le D_f(u, x_0). \end{aligned}$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in F$. Therefore by Proposition 4 the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded too, as claimed.

Now we will prove that each sequence $\{y_n^i\}_{n \in \mathbb{N}}$, $i = 1, 2, \ldots, N$, is bounded. Let $u \in F$. From the three point identity (see (2.3)) we get

$$\begin{aligned} D_f(u, x_n + e_n) &= D_f(u, x_n) - D_f(x_n + e_n, x_n) \\ &\quad + \langle \nabla f(x_n + e_n) - \nabla f(x_n), u - (x_n + e_n) \rangle \\ &\le D_f(u, x_n) + \langle \nabla f(x_n + e_n) - \nabla f(x_n), u - (x_n + e_n) \rangle. \end{aligned} \tag{3.4}$$

We also have

$$D_f(u, x_n) = D_f\left(u, \mathrm{proj}^f_{C_{n-1} \cap Q_{n-1}}(x_0)\right) \le D_f(u, x_0)$$

because the Bregman projection is QBNE and $F \subset C_{n-1} \bigcap Q_{n-1}$. On the other hand, since f is uniformly Fréchet differentiable and bounded on bounded subsets of X^*, we obtain from Proposition 1(ii) that

$$\lim_{n \to \infty} \|\nabla f(x_n + e_n) - \nabla f(x_n)\|_* = 0$$

because $\lim_{n \to \infty} e_n = 0$. This means that if we take into account that $\{x_n\}_{n \in \mathbb{N}}$ is bounded, then we get

$$\lim_{n \to \infty} \langle \nabla f(x_n) - \nabla f(x_n + e_n), u - (x_n + e_n) \rangle = 0. \tag{3.5}$$

Combining these facts, we obtain that $\{D_f(u, x_n + e_n)\}_{n \in \mathbb{N}}$ is bounded. Using the inequality

$$D_f\left(u, y_n^i\right) \le D_f(u, x_n + e_n),$$

we see that $\left\{D_f\left(u, y_n^i\right)\right\}_{n \in \mathbb{N}}$ is bounded too. The boundedness of the sequence $\{y_n^i\}_{n \in \mathbb{N}}$ now follows from Proposition 5. □

LEMMA 3. *For any* $i = 1, 2, \ldots, N$, *we have the following facts:*

(i)

$$\lim_{n \to \infty} \left[y_n^i - \left(x_n + e_n^i\right)\right] = 0; \tag{3.6}$$

(ii)

$$\lim_{n\to\infty}\left[\nabla f\left(y_n^i\right)-\nabla f\left(x_n+e_n^i\right)\right]=0; \tag{3.7}$$

(iii)

$$\lim_{n\to\infty}\left[f\left(y_n^i\right)-f\left(x_n+e_n^i\right)\right]=0. \tag{3.8}$$

PROOF. Since $x_{n+1}\in Q_n$ and $\operatorname{proj}^f_{Q_n}(x_0)=x_n$, it follows from Proposition 6(iii) that

$$D_f\left(x_{n+1},\operatorname{proj}^f_{Q_n}(x_0)\right)+D_f\left(\operatorname{proj}^f_{Q_n}(x_0),x_0\right)\le D_f\left(x_{n+1},x_0\right)$$

and hence

$$D_f\left(x_{n+1},x_n\right)+D_f\left(x_n,x_0\right)\le D_f\left(x_{n+1},x_0\right). \tag{3.9}$$

Therefore the sequence $\left\{D_f\left(x_n,x_0\right)\right\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Lemma 2), $\lim_{n\to\infty}D_f\left(x_n,x_0\right)$ exists. Thus from (3.9) it follows that

$$\lim_{n\to\infty}D_f\left(x_{n+1},x_n\right)=0. \tag{3.10}$$

Proposition 2 now implies that $\lim_{n\to\infty}\left(x_{n+1}-x_n\right)=0$. For any $i=1,2,\ldots,N$, it follows from the definition of the Bregman distance (see (2.2)) that

$$\begin{aligned}D_f\left(x_n,x_n+e_n^i\right)=f\left(x_n\right)-f\left(x_n+e_n^i\right)-\left\langle\nabla f\left(x_n+e_n^i\right),x_n-\left(x_n+e_n^i\right)\right\rangle=\\ f\left(x_n\right)-f\left(x_n+e_n^i\right)+\left\langle\nabla f\left(x_n+e_n^i\right),e_n^i\right\rangle.\end{aligned}$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [**14**, Proposition 1.1.11, p. 17]). In addition, f is uniformly Fréchet differentiable and therefore f is uniformly continuous on bounded subsets (see Proposition 1(i)). Hence, since $\lim_{n\to\infty}e_n^i=0$, it follows that

$$\lim_{n\to\infty}D_f\left(x_n,x_n+e_n^i\right)=0. \tag{3.11}$$

For any $i=1,2,\ldots,N$, it follows from the three point identity (see (2.3)) that

$$\begin{aligned}D_f\left(x_{n+1},x_n+e_n^i\right)&=D_f\left(x_{n+1},x_n\right)+D_f\left(x_n,x_n+e_n^i\right)\\&\quad+\left\langle\nabla f\left(x_n\right)-\nabla f\left(x_n+e_n^i\right),x_{n+1}-x_n\right\rangle.\end{aligned}$$

Since $\lim_{n\to+\infty}\left(x_{n+1}-x_n\right)=0$ and ∇f is bounded on bounded subsets of X, (3.10) and (3.11) imply that

$$\lim_{n\to\infty}D_f\left(x_{n+1},x_n+e_n^i\right)=0.$$

For any $i=1,2,\ldots,N$, it follows from the inclusion $x_{n+1}\in C_n^i$ that

$$D_f\left(x_{n+1},y_n^i\right)\le D_f\left(x_{n+1},x_n+e_n^i\right).$$

Hence $\lim_{n\to\infty}D_f\left(x_{n+1},y_n^i\right)=0$. Since $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded (see Lemma 2), Proposition 2 now implies that $\lim_{n\to\infty}\left(y_n^i-x_{n+1}\right)=0$. Therefore, for any $i=1,2,\ldots,N$, we have

$$\left\|y_n^i-x_n\right\|\le\left\|y_n^i-x_{n+1}\right\|+\left\|x_{n+1}-x_n\right\|\to 0.$$

Since $\lim_{n\to\infty}e_n^i=0$, it also follows that

$$\lim_{n\to\infty}\left[y_n^i-\left(x_n+e_n^i\right)\right]=0.$$

Since f is a uniformly Fréchet differentiable function and bounded on bounded subsets of X^*, it follows from Proposition 1(ii) that

$$\lim_{n\to\infty}\left[\nabla f\left(y_n^i\right)-\nabla f\left(x_n+e_n^i\right)\right]=0$$

for any $i=1,2,\ldots,N$. Finally, since f is uniformly Fréchet differentiable, it is also uniformly continuous on bounded subsets (see Proposition 1(i)) and therefore

$$\lim_{n\to\infty}\left[f\left(y_n^i\right)-f\left(x_n+e_n^i\right)\right]=0$$

for any $i=1,2,\ldots,N$. □

LEMMA 4. *If any weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to F, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_F^f(x_0)$.*

PROOF. From [**36**, Lemma 15.5, p.305] it follows that $F\left(T_n^i\right)$ is closed and convex for each $i=1,2,\ldots,N$ and $n\in\mathbb{N}$. Therefore F is nonempty, closed and convex, and the Bregman projection proj_F^f is well defined. Let $\tilde{u}=\operatorname{proj}_F^f(x_0)$. Since $x_{n+1}=\operatorname{proj}_{C_n\cap Q_n}^f(x_0)$ and F is contained in $C_n\bigcap Q_n$, we have $D_f(x_{n+1},x_0)\leq D_f(\tilde{u},x_0)$. Therefore Proposition 7 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u}=\operatorname{proj}_F^f(x_0)$, as claimed. □

Now we are ready to state and prove our main results. We begin with the first algorithm (Algorithm (3.1)).

THEOREM 1. *Let $\varepsilon>0$ and let K_i, $i=1,2,\ldots,N$, be N nonempty, closed and convex subsets of X such that $K:=\bigcap_{i=1}^N K_i$. Let $A_i:X\to X^*$, $i=1,2,\ldots,N$, be N BISM mappings such that $B(K_i,\varepsilon)\subset\operatorname{dom}A_i$ and $Z:=\bigcap_{i=1}^N\left(A_i^{-1}(0^*)\cap K_i\right)\neq\emptyset$. Let $f:X\to\mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. If, for each $i=1,2,\ldots,N$, the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}}\subset X$ satisfies $\left\|e_n^i\right\|<\varepsilon$ and $\lim_{n\to\infty}e_n^i=0$, then for each $x_0\in K$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1). Each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly as $n\to\infty$ to $\operatorname{proj}_V^f(x_0)$, where $V:=\bigcap_{i=1}^N VI(K_i,A_i)$.*

PROOF. We know that $\operatorname{dom}A_i^f=(\operatorname{dom}A_i)\bigcap(\operatorname{int}\operatorname{dom}f)=\operatorname{dom}A_i$ which implies that $B(K_i,\varepsilon)\subset\operatorname{dom}A_i^f$ for any $i=1,2,\ldots,N$. From Proposition 11 it follows that each A_i^f is a BFNE and therefore a QBNE operator with $F\left(A_i^f\right)=A_i^{-1}(0^*)$ for any $i=1,2,\ldots,N$. Thus $F\left(A_i^f\right)\supset A_i^{-1}(0^*)\bigcap K_i$.

Hence the set F from Condition 1 contains Z and therefore is nonempty. Denoting $T_n^i=A_i^f$ for any $i=1,2,\ldots,N$ and for each $n\in\mathbb{N}$, we see that Condition 1 holds and therefore we can apply our lemmata.

By Lemmata 1 and 2, any sequence $\{x_n\}_{n\in\mathbb{N}}$ which is generated by Algorithm (3.1) is well defined and bounded. From now on we let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence which is generated by Algorithm (3.1).

We claim that every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to V. From Lemma 3 we have

$$\begin{aligned}\lim_{n\to\infty}\left[y_n^i-\left(x_n+e_n^i\right)\right]&=\lim_{n\to\infty}\left[T_n^i\left(x_n+e_n^i\right)-\left(x_n+e_n^i\right)\right]\\&=\lim_{n\to\infty}\left[A_i^f\left(x_n+e_n^i\right)-\left(x_n+e_n^i\right)\right]=0\end{aligned}\tag{3.12}$$

for any $i = 1, 2, \ldots, N$. Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Let $z_n^i = x_n + e_n^i$. Since $x_{n_k} \rightharpoonup v$ and $e_{n_k}^i \to 0$, it is obvious that for any $i = 1, 2, \ldots, N$, the sequence $\left\{z_{n_k}^i\right\}_{k\in\mathbb{N}}$ converges weakly to v. We also have $\lim_{k\to\infty}\left(A_i^f z_{n_k}^i - z_{n_k}^i\right) = 0$ by (3.12). This means that $v \in \widehat{F}\left(A_i^f\right)\bigcap K_i$. Since each A_i^f is a BFNE operator (see Proposition 11(ii)), it follows from Propositions 10, 11(i) and 13 that $v \in \widehat{F}\left(A_i^f\right)\bigcap K_i = F\left(A_i^f\right)\bigcap K_i = A_i^{-1}(0^*)\bigcap K_i = VI(K_i, A_i)$ for any $i = 1, 2, \ldots, N$. Therefore $v \in V$, as claimed.

Now Theorem 1 is seen to follow from Lemma 4. □

In the next theorem we prove that Algorithm (3.2) also converges to a solution of a system of variational inequalities corresponding to a finite number of BISM mappings.

THEOREM 2. *Let the hypotheses of Theorem 1 hold. Then for each $x_0 \in K$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.2). Each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly as $n \to \infty$ to $\mathrm{proj}_V^f(x_0)$.*

PROOF. We know that $\mathrm{dom}\, A_i^f = (\mathrm{dom}\, A_i)\bigcap(\mathrm{int}\,\mathrm{dom}\, f) = \mathrm{dom}\, A_i$, which implies that $B(K_i, \varepsilon) \subset \mathrm{dom}\, A_i^f$ for any $i = 1, 2, \ldots, N$. From Proposition 11(ii) it follows that each A_i^f is a BFNE, hence a BSNE operator with $VI(K_i, A_i) = A_i^{-1}(0^*)\bigcap K_i \subset F\left(A_i^f\right) = \widehat{F}\left(A_i^f\right)$ for any $i = 1, 2, \ldots, N$ (see Propositions 10, 13 and Remark 4). We also know that the Bregman projection $\mathrm{proj}_{K_i}^f$ is a BFNE and therefore a BSNE operator with $F\left(\mathrm{proj}_{K_i}^f\right) = \widehat{F}\left(\mathrm{proj}_{K_i}^f\right)$ (see Remark 4). From Proposition 9 and Remark 3 we obtain that $\mathrm{proj}_{K_i}^f \circ A_i^f$ is a BSNE operator with $F\left(\mathrm{proj}_{K_i}^f \circ A_i^f\right) = \widehat{F}\left(\mathrm{proj}_{K_i}^f \circ A_i^f\right)$. Therefore $\mathrm{proj}_{K_i}^f \circ A_i^f$ is a QBNE operator (see Remark 4) with

$$\begin{aligned} F\left(\mathrm{proj}_{K_i}^f \circ A_i^f\right) &= F\left(\mathrm{proj}_{K_i}^f\right)\bigcap F\left(A_i^f\right) \\ &= K_i \bigcap A_i^{-1}(0^*) = VI(K_i, A_i). \end{aligned}$$

Hence the set F from Condition 1 is equal to Z and therefore nonempty. Denoting $T_n^i = \mathrm{proj}_{K_i}^f \circ A_i^f$ for any $i = 1, 2, \ldots, N$ and for each $n \in \mathbb{N}$, we see that Condition 1 holds and therefore we can apply our lemmata.

By Lemmata 1 and 2, any sequence $\{x_n\}_{n\in\mathbb{N}}$ which is generated by Algorithm (3.2) is well defined and bounded. From now on we let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence generated by Algorithm (3.2).

We claim that every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to V. Indeed, let $u \in V$. From the definition of the Bregman distance (see (2.2)) we obtain

$$\begin{aligned}
D_f\left(u, x_n + e_n^i\right) - D_f\left(u, y_n^i\right) &= \left[f(u) - f\left(x_n + e_n^i\right)\right.\\
&\quad \left.- \left\langle \nabla f\left(x_n + e_n^i\right), u - \left(x_n + e_n^i\right)\right\rangle\right]\\
&\quad - \left[f(u) - f\left(y_n^i\right) - \left\langle \nabla f\left(y_n^i\right), u - y_n^i\right\rangle\right]\\
&= f\left(y_n^i\right) - f\left(x_n + e_n^i\right) + \left\langle \nabla f\left(y_n^i\right), u - y_n^i\right\rangle\\
&\quad - \left\langle \nabla f\left(x_n + e_n^i\right), u - \left(x_n + e_n^i\right)\right\rangle\\
&= f\left(y_n^i\right) - f\left(x_n + e_n^i\right) + \left\langle \nabla f\left(y_n^i\right), x_n + e_n^i - y_n^i\right\rangle\\
&\quad + \left\langle \nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right), u - \left(x_n + e_n^i\right)\right\rangle. \qquad (3.13)
\end{aligned}$$

From Lemma 2 it follows that the sequence $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded and therefore $\left\{\nabla f\left(y_n^i\right)\right\}_{n\in\mathbb{N}}$ is bounded too. Thus from (3.6), (3.7), (3.8) and (3.13) we obtain that

$$\lim_{n\to\infty}\left[D_f\left(u, x_n + e_n^i\right) - D_f\left(u, y_n^i\right)\right] = 0.$$

From Propositions 6(iii) and 11(ii) we get

$$\begin{aligned}
D_f\left(u, y_n^i\right) &\le D_f\left(u, y_n^i\right) + D_f\left(y_n^i, A_i^f\left(x_n + e_n^i\right)\right) \le D_f\left(u, A_i^f\left(x_n + e_n^i\right)\right)\\
&\le D_f\left(u, x_n + e_n^i\right)
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} D_f\left(y_n^i, A_i^f\left(x_n + e_n^i\right)\right) = 0.$$

Proposition 2 now implies that

$$\lim_{n\to\infty}\left(y_n^i - A_i^f\left(x_n + e_n^i\right)\right) = 0.$$

Therefore

$$\left\|A_i^f\left(x_n + e_n^i\right) - x_n\right\| \le \left\|A_i^f\left(x_n + e_n^i\right) - y_n^i\right\| + \left\|y_n^i - x_n\right\| \to 0.$$

Since $\lim_{n\to\infty} e_n^i = 0$, we also obtain that

$$\lim_{n\to\infty}\left(A_i^f\left(x_n + e_n^i\right) - \left(x_n + e_n^i\right)\right) = 0 \qquad (3.14)$$

for any $i = 1, 2, \ldots, N$. Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Let $z_n^i = x_n + e_n^i$. Since $x_{n_k} \rightharpoonup v$ and $e_{n_k}^i \to 0$, it is obvious that for any $i = 1, 2, \ldots, N$, the sequence $\left\{z_{n_k}^i\right\}_{k\in\mathbb{N}}$ converges weakly to v. We also have $\lim_{k\to\infty}\left(A_i^f z_{n_k}^i - z_{n_k}^i\right) = 0$ by (3.14). This means that $v \in \widehat{F}\left(A_i^f\right)\bigcap K_i$. Since each A_i^f is a BFNE operator (see Proposition 11(ii)), it follows from Propositions 10 and 11(i) that $v \in \widehat{F}\left(A_i^f\right)\bigcap K_i = F\left(A_i^f\right)\bigcap K_i = A_i^{-1}(0^*)\bigcap K_i = VI(K_i, A_i)$ for any $i = 1, 2, \ldots, N$. Therefore $v \in V$, as claimed.

Now Theorem 2 is seen to follow from Lemma 4. □

Remark 5. *In this paper we solve the variational inequality problem for three different types of mappings. For the class of (single-valued) BISM mappings, the two problems of solving variational inequalities and finding zeroes are equivalent (see Proposition 13). Therefore there seems to be no reason to use Algorithm (3.2)* *instead of Algorithm (3.1) in this case, since Algorithm (3.2) is more complicated because of the presence of an additional projection. The usefulness and importance of Algorithm (3.2) comes into play when one wishes to solve a variational inequality problem corresponding to a class of mappings for which it is more general than the problem of finding zeroes. In this case one should use Algorithm (3.2) because of Proposition 12 (Algorithm (3.1) will not apply in this case). Also, in the next section (see Section 4) we deal with a different class of mappings, namely the pseudomonotone mappings, and there one must use Algorithm (3.2) in order to solve systems of variational inequalities corresponding to such mappings (see Theorem 3). In this connection, we now present an example where Algorithm (3.1) is not well-defined, but Algorithm (3.2) is and converges.*

Concerning Theorems 1 and 2, one may wonder whether the assumption $V = \bigcap_{i=1}^{N} VI\left(K_i, A_i\right) \neq \emptyset$ instead of $Z = \bigcap_{i=1}^{N}\left(A_i^{-1}\left(0^*\right) \cap K_i\right) \neq \emptyset$ would be sufficient. In the following example this condition is indeed sufficient for Algorithm (3.2), but not for Algorithm (3.1). It remains an open question whether this is always true.

EXAMPLE 3. *Take $N = 1$, $K_i = K$, X, f and $A_1 = A$ as in Example 2 and let $\epsilon > 0$ be arbitrary. Thus $V = \{1\} \neq \emptyset$. Let $e_n^1 = 0$ for all n. Then all the assumptions of Theorem 1 are satisfied when the assumption that $Z \neq \emptyset$ is replaced with $V \neq \emptyset$. However, for $1 \leq x_0 < 2$ one gets $y_0^1 = 0$ (note that A_1^f is the zero operator in our case) and*

$$C_0^1 = \left\{z \in K : z^2 \leq \left(z - x_0\right)^2\right\} = \left\{z \geq 1 : z \leq \frac{x_0}{2} < 1\right\} = \emptyset.$$

Therefore Algorithm (3.1) is not well defined. This means that $V \neq \emptyset$ is not sufficient for Theorem 1.

On the other hand, in the case of Algorithm (3.2) we still have $A_1^f = 0$, but $y_n^1 = 1$ for all $n \in \mathbb{N}$. Therefore the set C_0^1 is nonempty. More precisely,

$$C_0^1 = \left\{z \in K : (z-1)^2 \leq \left(z - x_0\right)^2\right\} = \left\{z \geq 1 : z \leq \frac{x_0 + 1}{2}\right\} = \left[1, \frac{x_0 + 1}{2}\right],$$

i.e., $C_0^1 = \{1\}$ when $x_0 = 1$ and is a proper closed interval for $x_0 > 1$. We distinguish two cases:

Case 1: $x_0 = 1$. *We have $C_n^1 = Q_n = K$ for all $n \in \mathbb{N}$, so that $x_n = x_0 = 1$ (a constant sequence) and Algorithm (3.2) converges to the (unique) solution of the corresponding variational inequality.*

Case 2: $x_0 > 1$. *It can be easily shown (by induction) that*

$$C_n^i = [1, (1/2)\left(x_n + 1\right)] \subset Q_n = [1, x_n]$$

and $x_{n+1} = (1/2)\left(x_n + 1\right)$. Since the sequence $\left\{x_n\right\}_{n \in \mathbb{N}}$ is strictly decreasing, it follows that its limit is again 1, the (unique) solution of the corresponding variational inequality.

The final conclusion is that Algorithm (3.2) generates a sequence which (strongly) converges to $\operatorname{proj}_V^f\left(x_0\right)$.

From Proposition 13 we know that the problem of solving variational inequalities on K and the problem of finding zeroes of BISM mappings in K are one and the same. Therefore we can use (directly) Algorithms (3.1) and (3.2) to approximate common zeroes of finitely many Bregman inverse strongly monotone mappings.

REMARK 6. *As for possible implementations of Algorithm (3.1) and (3.2), note that as we have already observed, each $C_n \cap Q_n$ is a closed polyhedral set and therefore computing the projection of the starting point x_0 onto it is not that difficult, at least in the case where the space X is a Hilbert space and $f = (1/2)\left\|\cdot\right\|^2$.*

4. Solving Variational Inequalities for Pseudomonotone Mappings

In this section we show that our Algorithm (3.2) can also be implemented to solve systems of variational inequalities for another class of mappings of monotone type (in this connection see also Remark 5). If the variational inequalities correspond to BISM mappings, then we are in the setting of Section 3. If the mappings to which the variational inequalities correspond are not BISM, then the situation is more complicated.

As we already know, when A_i, $i = 1, 2, \ldots, N$, *are* (single-valued) BISM mappings, the assumption $Z := \bigcap_{i=1}^{N} A_i^{-1}(0^*) \bigcap K_i \neq \emptyset$ leads to $Z = \bigcap_{i=1}^{N} VI(K_i, A_i)$ (see Proposition 13). When the mappings A_i, $i = 1, 2, \ldots, N$, are *not* BISM, it is well known that the system of variational inequalities might have solutions even when there are no common zeroes. Hence we will assume that $V := \bigcap_{i=1}^{N} VI(K_i, A_i) \neq \emptyset$, but not that $\bigcap_{i=1}^{N}\left(A_i^{-1}(0^*) \bigcap K_i\right) \neq \emptyset$.

Our next result shows that Algorithm (3.2) solves systems of variational inequalities for pseudomonotone mappings.

THEOREM 3. *Let $\varepsilon > 0$ and let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to X^*$, $i = 1, 2, \ldots, N$, be N pseudomonotone mappings which are bounded on bounded subsets of $B(K_i, \varepsilon)$ such that $B(K_i, \varepsilon) \subset \operatorname{dom} A_i$ and $V := \bigcap_{i=1}^{N} VI(K_i, A_i) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. Assume that each A_i^f, $i = 1, 2, \ldots, N$, is BSNE. If, for each $i = 1, 2, \ldots, N$, the sequence of errors $\left\{e_n^i\right\}_{n \in \mathbb{N}} \subset X$ satisfies $\left\|e_n^i\right\| < \varepsilon$ and $\lim_{n \to \infty} e_n^i = 0$, then for each $x_0 \in K$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (3.2). Each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly as $n \to \infty$ to $\operatorname{proj}_V^f(x_0)$.*

PROOF. We know that $\operatorname{dom} A_i^f = (\operatorname{dom} A_i) \cap (\operatorname{int} \operatorname{dom} f) = \operatorname{dom} A_i$, which implies that $B(K_i, \varepsilon) \subset \operatorname{dom} A_i^f$ for any $i = 1, 2, \ldots, N$. By assumption, each A_i^f is a BSNE operator with $F\left(A_i^f\right) = \widehat{F}\left(A_i^f\right)$ for any $n \in \mathbb{N}$ (see Proposition 10). We also know that the Bregman projection $\operatorname{proj}_{K_i}^f$ is a BFNE and therefore a BSNE operator with $F\left(\operatorname{proj}_{K_i}^f\right) = \widehat{F}\left(\operatorname{proj}_{K_i}^f\right)$ (see Remark 4). From Remark 3 we obtain that $\operatorname{proj}_{K_i}^f \circ A_i^f$ is a BSNE operator with $F\left(\operatorname{proj}_{K_i}^f \circ A_i^f\right) = \widehat{F}\left(\operatorname{proj}_{K_i}^f \circ A_i^f\right)$. Therefore $\operatorname{proj}_{K_i}^f \circ A_i^f$ is a QBNE operator (see Remark 4) and from Proposition 12 we also have

$$F\left(\operatorname{proj}_{K_i}^f \circ A_i^f\right) = VI(K_i, A_i).$$

Hence the set F from Condition 1 is equal to V and therefore is nonempty, closed and convex (see [**36**, Lemma 15.5, p. 305]). Denoting $T_n^i = \operatorname{proj}_{K_i}^f \circ A_i^f$ for any $i = 1, 2, \ldots, N$, we see that Condition 1 holds and therefore we may apply our lemmata.

By Lemmata 1 and 2, any sequence $\{x_n\}_{n\in\mathbb{N}}$ which is generated by Algorithm (3.2) is well defined and bounded. From now on we let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence generated by Algorithm (3.2).

We claim that every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to V. Indeed, since $y_n^i = \operatorname{proj}_{K_i}^f \left(A_i^f \left(x_n + e_n^i\right)\right)$, we know by Proposition 6(ii) that

$$\left\langle \nabla f \left(A_i^f \left(x_n + e_n^i\right)\right) - \nabla f \left(y_n^i\right), y_n^i - y \right\rangle \geq 0$$

for any $y \in K_i$ and for all $i = 1, 2, \ldots, N$, which yields

$$\left\langle \nabla f \left(x_n + e_n^i\right) - A_i \left(x_n + e_n^i\right) - \nabla f \left(y_n^i\right), y_n^i - y \right\rangle \geq 0 \tag{4.1}$$

for any $y \in K_i$ and for all $i = 1, 2, \ldots, N$. From Lemma 2 it follows that the sequence $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded. Thus from (3.7) we obtain that

$$\lim_{n\to\infty} \left\langle \nabla f \left(x_n + e_n^i\right) - \nabla f \left(y_n^i\right), y_n^i - y \right\rangle = 0$$

and this leads by (4.1) to

$$\liminf_{n\to\infty} \left\langle -A_i \left(x_n + e_n^i\right), y_n^i - y \right\rangle \geq 0$$

or, equivalently, to

$$\limsup_{n\to\infty} \left\langle A_i \left(x_n + e_n^i\right), y_n^i - y \right\rangle \leq 0 \tag{4.2}$$

for any $y \in K_i$ and for all $i = 1, 2, \ldots, N$. On the other hand,

$$\left\langle A_i \left(x_n + e_n^i\right), y_n^i - y \right\rangle = \left\langle A_i \left(x_n + e_n^i\right), x_n + e_n^i - y \right\rangle + \left\langle A_i \left(x_n + e_n^i\right), y_n^i - x_n - e_n^i \right\rangle. \tag{4.3}$$

Since the sequence $\left\{x_n + e_n^i\right\}_{n\in\mathbb{N}}$ is bounded, it follows that the sequence $\left\{A_i \left(x_n + e_n^i\right)\right\}_{n\in\mathbb{N}}$ is also bounded because A_i is bounded on bounded subsets of $B\left(K_i, \varepsilon\right)$, and this implies, when combined with (3.6), that the second term on the right-hand side of (4.3) converges to zero. Thus from (4.2) we see that

$$\limsup_{n\to\infty} \left\langle A_i \left(x_n + e_n^i\right), x_n + e_n^i - y \right\rangle \leq 0 \tag{4.4}$$

for any $y \in K_i$ and for all $i = 1, 2, \ldots, N$.

Now let $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Denoting its weak limit by v, we observe that the sequence $\left\{x_{n_j} + e_{n_j}^i\right\}_{j\in\mathbb{N}}$ also converges weakly to v. From (4.4) we obtain that

$$\limsup_{j\to\infty} \left\langle A_i \left(x_{n_j} + e_{n_j}^i\right), x_{n_j} + e_{n_j}^i - v \right\rangle \leq 0 \tag{4.5}$$

for all $i = 1, 2, \ldots, N$. Since each A_i is pseudomonotone, we obtain from (4.4) and (4.5) that

$$\left\langle A_i v, v - y \right\rangle \leq \liminf_{j\to\infty} \left\langle A_i \left(x_{n_j} + e_{n_j}^i\right), x_{n_j} + e_{n_j}^i - y \right\rangle \leq 0$$

for any $y \in K_i$ and for all $i = 1, 2, \ldots, N$. Thus $v \in VI\left(K_i, A_i\right)$ for each $i = 1, 2, \ldots, N$ and so $v \in V$, as claimed.

Now we see that Theorem 3 follows from Lemma 4. $\square$

5. Solving Variational Inequalities for Hemicontinuous Mappings

In this section we present a method for solving systems of variational inequalities for hemicontinuous mappings. One way to do this is to use the following result. Consider the *normal cone* N_K corresponding to $K \subset X$, which is defined by

$$N_K(x) := \{\xi \in X^* : \langle \xi, x - y\rangle \geq 0, \forall y \in K\}, \ x \in K.$$

PROPOSITION 14 (*cf.* [**37**, Theorem 3, p. 77]). *Let K be a nonempty, closed and convex subset of X, and let $A : K \to X^*$ be a monotone and hemicontinuous mapping. Let $B : X \to 2^{X^*}$ be the mapping which is defined by*

$$Bx := \begin{cases} (A + N_K)\,x, & x \in K \\ \emptyset, & x \notin K. \end{cases} \tag{5.1}$$

Then B is maximal monotone and $B^{-1}(0^*) = VI(K, A)$.

For each $i = 1, 2, \ldots, N$, let the operator B_i, defined as in (5.1), correspond to the mapping A_i and the set K_i, and let $\left\{\lambda_n^i\right\}_{n\in\mathbb{N}}$, $i = 1, 2, \ldots, N$, be N sequences of positive real numbers.

The authors of [**34**] considered the following algorithm for finding common zeroes of finitely many maximal monotone mappings. More precisely, they introduced there the following algorithm:

$$\begin{cases} x_0 \in X, \\ y_n^i = \operatorname{Res}^f_{\lambda_n^i B_i}\left(x_n + e_n^i\right), \\ C_n^i = \left\{z \in X : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\right\}, \\ x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n}(x_0),\ n = 0, 1, 2, \ldots, \end{cases} \tag{5.2}$$

and obtained the following result.

PROPOSITION 15 (*cf.* [**34**, Theorem 4.2, p. 35]). *Let $B_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone operators such that $Z := \bigcap_{i=1}^N B_i^{-1}(0^*) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (5.2). If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, and the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly as $n \to \infty$ to $\operatorname{proj}^f_Z(x_0)$.*

This result yields a method for solving systems of variational inequalities corresponding to hemicontinuous mappings.

THEOREM 4. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^N K_i$. Let $A_i : K_i \to X^*$, $i = 1, 2, \ldots, N$, be N monotone and hemicontinuous mappings with $V := \bigcap_{i=1}^N VI(K_i, A_i) \neq \emptyset$. Let $\left\{\lambda_n^i\right\}_{n\in\mathbb{N}}$, $i = 1, 2, \ldots, N$, be N sequences of positive real numbers that satisfy $\liminf_{n\to\infty} \lambda_n^i > 0$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. If, for each $i = 1, 2, \ldots, N$, the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to\infty} e_n^i = 0$, then for each $x_0 \in K$, there are* *sequences* $\{x_n\}_{n\in\mathbb{N}}$ *which satisfy (5.2), where each* B_i *is defined as in (5.1). Each such sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly as* $n\to\infty$ *to* $\operatorname{proj}_V^f(x_0)$.

Proof. For each $i=1,2,\ldots,N$, we define the mapping B_i as in (5.1). Proposition 14 now implies that each B_i, $i=1,2,\ldots,N$, is a maximal monotone mapping and $V=\bigcap_{i=1}^N VI(K_i,A_i)=\bigcap_{i=1}^N B_i^{-1}(0^*)\neq\emptyset$.

Our result now follows immediately from Proposition 15 with $Z=V$. □

Now we present another way for solving systems of variational inequalities corresponding to hemicontinuous mappings. To this end, we will need the following notions.

Let K be a closed and convex subset of X, and let $g:K\times K\to\mathbb{R}$ be a bifunction satisfying the following conditions:

(C1) $g(x,x)=0$ for all $x\in K$;
(C2) g is monotone, *i.e.*, $g(x,y)+g(y,x)\leq 0$ for all $x,y\in K$;
(C3) for all $x,y,z\in K$,

$$\limsup_{t\downarrow 0} g(tz+(1-t)x,y)\leq g(x,y);$$

(C4) for each $x\in K$, $g(x,\cdot)$ is convex and lower semicontinuous.

The *equilibrium problem* corresponding to g is to find $\bar{x}\in K$ such that

$$g(\bar{x},y)\geq 0\ \ \forall y\in K. \tag{5.3}$$

The solutions set of (5.3) is denoted by $EP(g)$. For more information on this problem see, for instance, [**7, 22, 26, 27, 28**].

Proposition 16. *Let* $A:X\to X^*$ *be a monotone mapping such that* $K:=\operatorname{dom}A$ *is closed and convex. Assume that* A *is bounded on bounded subsets and hemicontinuous on* K*. Then the bifunction* $g(x,y)=\langle Ax,y-x\rangle$ *satisfies conditions (C1)–(C4).*

Proof. It is clear that $g(x,x)=\langle Ax,x-x\rangle=0$ for any $x\in K$. From the monotonicity of the mapping A we obtain that

$$g(x,y)+g(y,x)=\langle Ax,y-x\rangle+\langle Ay,x-y\rangle=\langle Ax-Ay,y-x\rangle\leq 0$$

for any $x,y\in K$. To prove (C3), fix $y\in X$ and choose the sequence $\{t_n\}_{n\in\mathbb{N}}$, converging to zero, such that

$$\limsup_{t\downarrow 0} g(tz+(1-t)x,y)=\lim_{n\to\infty} g(t_nz+(1-t_n)x,y).$$

Such a sequence exists by the definition of the limsup. Denote $u_n=t_nz+(1-t_n)x$. Then $\lim_{n\to\infty}u_n=x$ and $\{Au_n\}_{n\in\mathbb{N}}$ is bounded. Let $\{Au_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence. Then its limit is Ax because A is hemicontinuous and we get

$$\begin{aligned}\limsup_{t\downarrow 0} g(tz+(1-t)x,y)&=\lim_{k\to\infty} g(t_{n_k}z+(1-t_{n_k})x,y)=\\&=\lim_{k\to\infty}\langle A(t_{n_k}z+(1-t_{n_k})x),y-t_{n_k}z-(1-t_{n_k})x\rangle\\&=\lim_{k\to\infty}\langle A(u_{n_k}),y-u_{n_k}\rangle=\langle Ax,y-x\rangle=g(x,y)\end{aligned}$$

for all $x, y, z \in K$, as required. The last condition (C4) also holds because

$$\begin{aligned} g\left(x, ty_1 + (1-t)\, y_2\right) &= \left\langle Ax, x - \left(ty_1 + (1-t)\, y_2\right)\right\rangle \\ &= t\left\langle Ax, x - y_1\right\rangle + (1-t)\left\langle Ax, x - y_2\right\rangle \\ &= tg\left(x, y_1\right) + (1-t)\, g\left(x, y_2\right); \end{aligned}$$

thus the function $g(x, \cdot)$ is clearly convex and lower semicontinuous as it is (in particular) affine and continuous for any $x \in K$.

Therefore g indeed satisfies conditions (C1)–(C4). $\square$

The *resolvent of a bifunction* $g : K \times K \to \mathbb{R}$ is the operator $\mathrm{Res}_g^f : X \to 2^K$ defined by (see [**35**])

$$\mathrm{Res}_g^f(x) = \left\{z \in K : g(z, y) + \left\langle \nabla f(z) - \nabla f(x), y - z\right\rangle \geq 0 \ \ \forall y \in K\right\}.$$

Proposition 17 (*cf.* [**35**, Lemmata 1 and 2, pp. 130-131]). *Let $f : X \to (-\infty, +\infty]$ be a supercoercive Legendre function. Let K be a closed and convex subset of X. If the bifunction $g : K \times K \to \mathbb{R}$ satisfies conditions (C1)–(C4), then:*

(i) $\mathrm{dom}\left(\mathrm{Res}_g^f\right) = X$;
(ii) Res_g^f *is single-valued;*
(iii) Res_g^f *is a BFNE operator;*
(iv) *the set of fixed points of* Res_g^f *is the solution set of the corresponding equilibrium problem, i.e.,* $F\left(\mathrm{Res}_g^f\right) = EP(g)$;
(v) $EP(g)$ *is a closed and convex subset of* K.

Combining Propositions 17 and 16, we arrive at the following result.

Proposition 18. *Let $f : X \to (-\infty, +\infty]$ be a supercoercive Legendre function. Let $A : X \to X^*$ be a monotone mapping such that $K := \mathrm{dom}\, A$ is closed and convex. Assume that A is bounded on bounded subsets and hemicontinuous on K. Then the generalized resolvent of A, defined by*

$$\mathrm{GRes}_A^f(x) := \left\{z \in K : \left\langle Az, y - z\right\rangle + \left\langle \nabla f(z) - \nabla f(x), y - z\right\rangle \geq 0\, \forall y \in K\right\}, \tag{5.4}$$

has the following properties:

(i) $\mathrm{dom}\left(\mathrm{GRes}_A^f\right) = X$;
(ii) GRes_A^f *is single-valued;*
(iii) GRes_A^f *is a BFNE operator;*
(iv) *the set of fixed points of* GRes_A^f *is the solution set of the corresponding variational inequality problem, i.e.,* $F\left(\mathrm{GRes}_A^f\right) = VI(K, A)$;
(v) $VI(K, A)$ *is a closed and convex subset of* K.

The connection between the resolvent Res_A^f and the generalized resolvent GRes_A^f is brought out by the following remark.

Remark 7. *If the domain of the mapping A is the whole space, then $VI(X, A)$ is exactly the zero set of A. Therefore we obtain for $z \in \mathrm{GRes}_A^f(x)$ that*

$$\left\langle Az, y - z\right\rangle + \left\langle \nabla f(z) - \nabla f(x), y - z\right\rangle \geq 0$$

for any $y \in X$. This is equivalent to

$$\langle Az + \nabla f(z) - \nabla f(x), y - z \rangle \geq 0$$

for any $y \in X$, and this, in turn, is the same as

$$\langle Az + \nabla f(z) - \nabla f(x), w \rangle \geq 0$$

for any $w \in X$. But then we obtain that

$$\langle Az + \nabla f(z) - \nabla f(x), w \rangle = 0$$

for any $w \in X$. This happens only if $Az + \nabla f(z) - \nabla f(x) = 0^*$, which means that $z = (\nabla f + A)^{-1} \nabla f(x)$. This proves that the generalized resolvent GRes_A^f is a generalization of the resolvent Res_A^f.

Now we are ready to present another algorithm for solving systems of variational inequalities. More precisely, we consider the following algorithm:

$$(5.5) \qquad \begin{cases} x_0 \in X, \\ y_n^i = \mathrm{GRes}_{\lambda_n^i A_i}^f \left(x_n + e_n^i \right), \\ C_n^i = \left\{ z \in K_i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right) \right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{ z \in K : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0 \right\}, \\ x_{n+1} = \mathrm{proj}_{C_n \cap Q_n}^f (x_0), \; n = 0, 1, 2, \ldots. \end{cases}$$

THEOREM 5. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^N K_i$. Let $A_i : K_i \to X^*$, $i = 1, 2, \ldots, N$, be N monotone and hemicontinuous mappings and assume that $V := \bigcap_{i=1}^N VI(K_i, A_i) \neq \emptyset$. Let $\left\{\lambda_n^i\right\}_{n \in \mathbb{N}}$, $i = 1, 2, \ldots, N$, be N sequences of positive real numbers that satisfy $\liminf_{n \to \infty} \lambda_n^i > 0$. Let $f : X \to \mathbb{R}$ be a supercoercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. If, for each $i = 1, 2, \ldots, N$, the sequence of errors $\left\{e_n^i\right\}_{n \in \mathbb{N}} \subset X$ satisfies $\lim_{n \to \infty} e_n^i = 0$, then for each $x_0 \in K$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (5.5). Each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly as $n \to \infty$ to $\mathrm{proj}_V^f(x_0)$.*

PROOF. Denote $T_n^i = \mathrm{GRes}_{\lambda_n^i A_i}^f$ for any $i = 1, 2, \ldots, N$ and for each $n \in \mathbb{N}$. From Proposition 18 it follows that each $\mathrm{GRes}_{\lambda_n^i A_i}^f$ is a single-valued BFNE operator with full domain, and hence a QBNE operator (see Remark 4) with $F\left(\mathrm{GRes}_{\lambda_n^i A_i}^f\right) = VI(K_i, A_i)$ for each $i = 1, 2, \ldots, N$ and for any $n \in \mathbb{N}$. Hence the set F from Condition 1 (when $\varepsilon = 0$) is equal to V and therefore nonempty. Thus Condition 1 holds and we can use our lemmata.

By Lemmata 1 and 2, any sequence $\{x_n\}_{n \in \mathbb{N}}$ which is generated by (5.5) is well defined and bounded. From now on we let $\{x_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence generated by (5.5).

We claim that every weak subsequential limit of $\{x_n\}_{n \in \mathbb{N}}$ belongs to V. Indeed, by the definition of y_n^i we know that

$$\lambda_n^i \left\langle A_i y_n^i, y - y_n^i \right\rangle + \left\langle \nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right), y - y_n^i \right\rangle \geq 0$$

for all $y \in K_i$. Hence from the monotonicity of A it follows that

$$\left\langle \nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right), y - y_n^i\right\rangle \geq \lambda_n^i \left\langle A_i y_n^i, y_n^i - y\right\rangle \geq \lambda_n^i \left\langle A_i y, y_n^i - y\right\rangle \tag{5.6}$$

for all $y \in K_i$. Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Then from (3.6) we see that $\left\{y_{n_k}^i\right\}_{k\in\mathbb{N}}$ also converges weakly to v for any $i = 1, 2, \ldots, N$. Replacing n by n_k in (5.6), we get

$$\left\langle \nabla f\left(y_{n_k}^i\right) - \nabla f\left(x_{n_k} + e_{n_k}^i\right), y - y_{n_k}^i\right\rangle \geq \lambda_{n_k}^i \left\langle A_i y, y_{n_k}^i - y\right\rangle. \tag{5.7}$$

Since the sequence $\left\{y_{n_k}^i\right\}_{k\in\mathbb{N}}$ is bounded and $\liminf_{k\to\infty} \lambda_{n_k}^i > 0$, it follows from (3.7) and (5.7) that

$$\langle A_i y, y - v\rangle \geq 0, \tag{5.8}$$

for each $y \in K_i$ and for any $i = 1, 2, \ldots, N$. For any $t \in (0, 1]$, we now define $y_t = ty + (1-t)\, v$. Let $i = 1, 2, \ldots, N$. Since y and v belong to K_i, it follows from the convexity of K_i that $y_t \in K_i$ too. Hence $\langle A_i y_t, y_t - v\rangle \geq 0$ for any $i = 1, 2, \ldots, N$. Thus

$$0 = \langle A_i y_t, y_t - y_t\rangle = t\, \langle A_i y_t, y_t - y\rangle + (1-t)\, \langle A_i y_t, y_t - v\rangle \geq t\, \langle A_i y_t, y_t - y\rangle.$$

Dividing by t, we obtain that $\langle A_i y_t, y - y_t\rangle \geq 0$ for all $y \in K_i$.

Let $\{t_n\}_{n\in\mathbb{N}}$ be a positive sequence such that $\lim_{n\to\infty} t_n = 0$. Denote $y_n = y_{t_n}$ for each $n \in \mathbb{N}$. Since the mapping A is hemicontinuous we know that w-$\lim_{n\to\infty} A_i y_n = A_i v$. The sequence $\{A_i y_n\}_{n\in\mathbb{N}}$ is bounded as a weakly convergent sequence. Therefore

$$\lim_{n\to\infty} \langle A_i y_n, y - y_n\rangle = \lim_{n\to\infty} \left(\langle A_i y_n, v - y_n\rangle + \langle A_i y_n, y - v\rangle\right) = \langle A_i v, y - v\rangle.$$

Hence $\langle A_i v, y - v\rangle \geq 0$ for all $y \in K_i$. Thus $v \in VI\left(K_i, A_i\right)$ for any $i = 1, 2, \ldots, N$. Therefore $v \in V$, as claimed.

Now Theorem 5 is seen to follow from Lemma 4. □

We close this section with the following two open problems.

Problem 2. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X. Let $A : K \to X^*$ be a monotone and hemicontinuous mapping. Then the generalized resolvent $R := \mathrm{GRes}_A^f$ is a single-valued BFNE operator with full domain. From* [**6**, Proposition 5.1, p. 7] *we know that $S := \nabla f \circ R^{-1} - \nabla f$ is a maximal monotone mapping. What are the connections between A and S?*

Remark 8. *A connection between the mappings A and S exists, for example, when the operator R is taken to be the resolvent Res_A^f of the mapping A (cf. Remark 7). In this case $S = A$.*

Problem 3. *The above mapping S is maximal monotone. The mapping B (see 5.1) is a maximal monotone extension of the mapping A. What are the connections between B and S?*

6. Particular Cases

6.1. Uniformly Smooth and Uniformly Convex Banach Spaces. In this subsection we assume that X is a uniformly smooth and uniformly convex Banach space. We also assume that the function f is equal to $(1/2)\left\|\cdot\right\|^2$. It is well known that in this case $\nabla f = J$, where J is the *normalized duality mapping* of the space X. In this case the function f is Legendre (see [**3**, Lemma 6.2, p. 24]) and uniformly Fréchet differentiable on bounded subsets of X. According to [**15**, Corollary 1(ii), p. 325], f is sequentially consistent since X is uniformly convex and hence f is totally convex on bounded subsets of X. Therefore Theorems 1-5 hold in this context and improve upon previous results.

Our algorithms are more flexible than previous algorithms because they leave us the freedom of fitting the function f to the nature of the mapping A and of the space X in ways which make the application of these algorithms simpler. These computations can be simplified by an appropriate choice of the function f. For instance, if $X = \ell^p$ or $X = L^p$ with $p \in (1, +\infty)$, and $f(x) = (1/p)\|x\|^p$, then the computations become simpler than those required in other algorithms, which correspond to $f(x) = (1/2)\|x\|^2$. In this connection see, for instance, [**15**].

6.2. Hilbert Spaces. In this subsection we assume that X is a Hilbert space. We also assume that the function f is equal to $(1/2)\|\cdot\|^2$. It is well known that in this case $X = X^*$ and $\nabla f = I$, where I is the identity operator. Now we list our main notions under these assumptions.

(1) The Bregman distance $D_f(x, y)$ and the Bregman projection proj_K^f become $(1/2)\|x - y\|^2$ and the metric projection P_K, respectively.
(2) Both the classes of BISM mappings and BFNE operators become the class of firmly nonexpansive operators: recall that in this setting an operator $T : K \to K$ is called *firmly nonexpansive* if
$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$$
for any $x, y \in K$.
(3) The resolvent Res_A^f and the anti-resolvent A^f of a mapping A become the classical resolvent $R_A = (I + A)^{-1}$ and $I - A$, respectively.

Now our Algorithms (3.1) and (3.2) take the following form:

$$\begin{cases} x_0 \in K = \bigcap_{i=1}^N K_i, \\ y_n^i = (I - A_i)\left(x_n + e_n^i\right), \\ C_n^i = \left\{z \in K_i : \left\|z - y_n^i\right\| \le \left\|z - \left(x_n + e_n^i\right)\right\|\right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in K : \langle x_0 - x_n, z - x_n\rangle \le 0\right\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \; n = 0, 1, 2, \ldots, \end{cases} \tag{6.1}$$

and

$$\begin{cases} x_0 \in K = \bigcap_{i=1}^N K_i, \\ y_n^i = P_{K_i}(I - A_i)\left(x_n + e_n^i\right), \\ C_n^i = \left\{z \in K_i : \left\|z - y_n^i\right\| \le \left\|z - \left(x_n + e_n^i\right)\right\|\right\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in K : \langle x_0 - x_n, z - x_n\rangle \le 0\right\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \; n = 0, 1, 2, \ldots, \end{cases} \tag{6.2}$$

In this case Algorithms (6.1) and (6.2) solve systems of variational inequalities corresponding to firmly nonexpansive operators (see (2) above).

Another interesting case is where the function f is equal to $(1/2\alpha)\|\cdot\|^2$. Then the class of BISM mappings becomes the class of α-inverse strongly monotone operators. There are many papers that solve variational inequalities corresponding to this class of mappings. Most of them also assume that the α-inverse strongly monotone mapping A satisfies the following condition:

$$\|Ay\| \leq \|Ay - Au\|$$

for all $y \in K$ and $u \in VI(K, A)$ (see, for example, [**25**]). In our results this assumption is unnecessary. Hence our Algorithms (3.1) and (3.2) solve systems of variational inequalities corresponding to general α-inverse strongly monotone operators.

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