Two Strong Convergence Theorems for Bregman Strongly Nonexpansive Operators in Reflexive Banach Spaces

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Abstract. We study the convergence of two iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in reflexive Banach spaces. Both algorithms take into account possible computational errors. We establish two strong convergence theorems and then apply them to the solution of convex feasibility, variational inequality and equilibrium problems.

1. Introduction

Let X denote a real reflexive Banach space with norm $\|\cdot\|$ and let X^* stand for the (topological) dual of X equipped with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x \rangle$.

An operator $A : X \to 2^{X^*}$ is said to be *monotone* if for any $x, y \in \operatorname{dom} A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \quad \Longrightarrow \quad \langle \xi - \eta, x - y \rangle \geq 0.$$

(Recall that the set $\operatorname{dom} A = \{x \in X \ : \ Ax \neq \varnothing\}$ is called the *effective domain* of such an operator A.) A monotone operator A is said to be *maximal* if graph A, the graph of A, is not a proper subset of the graph of any other monotone operator. In this paper $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and

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convex function, and $f^* : X^* \to (-\infty, +\infty]$ is the Fenchel conjugate of f. The set of nonnegative integers will be denoted by $\mathbb{N}$.

Let K be a nonempty, closed and convex subset of a Hilbert space H. An operator $T : K \to K$ is said to be *nonexpansive* (or 1-Lipschitz) if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. It turns out that nonexpansive fixed point theory can be applied to the solution of diverse problems such as finding zeroes of monotone operators and solutions to certain evolution equations, and solving convex feasibility (CFP), variational inequality (VIP) and equilibrium problems (EP). In some cases it is enough to assume that the operator T is *quasi-nonexpansive*, that is, $\|p - Tx\| \leq \|p - x\|$ for all $p \in F(T)$ and $x \in K$, where $F(T)$ stands for the (nonempty) fixed point set of T. There are, in fact, many papers that deal with methods for finding fixed points of nonexpansive and quasi-nonexpansive operators in Hilbert space.

When we try to extend this theory to Banach spaces we encounter some difficulties because many of the useful examples of nonexpansive operators in Hilbert space are no longer nonexpansive in Banach spaces (for example, the resolvent $R_A = (I + A)^{-1}$ of a maximal monotone operator $A : H \to 2^H$ and the metric projection P_K onto a nonempty, closed and convex subset K of H). There are several ways to overcome these difficulties. One of them is to use the Bregman distance (see Section 2.3) instead of the norm and Bregman (quasi-) nonexpansive operators instead of (quasi-) nonexpansive operators (see Section 2.5 for more details). The Bregman projection (Section 2.4) and the generalized resolvent (Section 5) are examples of Bregman (quasi-) nonexpansive operators.

In this paper we are concerned with Bregman strongly nonexpansive operators (see Section 2.5). Our main goal is to study the convergence of two iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in reflexive Banach spaces. Both algorithms take into account possible computational errors. We establish two strong convergence theorems (Theorems 1 and 2 below) and then get as corollaries two methods for solving convex feasibility problems (Corollaries 1 and 2), finding zeroes of maximal monotone operators (Corollaries 3 and 4) and solving equilibrium problems (Corollaries 5 and 6). All these corollaries also allow for possible computational errors. In addition, we obtain two methods for finding zeroes of Bregman inverse strongly monotone operators (Corollaries 7 and 8) and for solving certain variational inequalities (Corollaries 9 and 10).

The paper is organized as follows. In Section 2 we present several preliminary definitions and results. The third section is devoted to the study of our two iterative methods. In Sections 4–8 we modify these methods in order to solve other problems: convex feasibility problems (Section 4), finding zeroes of maximal monotone operators (Section 5), equilibrium problems (Section 6), finding zeroes of Bregman inverse strongly monotone operators (Section 7) and variational inequalities (Section 8). For more information regarding these problems see, for example, [**2**], [**26**], [**6**], [**12**] and [**17, 18**], respectively. In Section 9 we observe that our methods may also be used for finding common solutions to mixed problems.

2. Preliminaries

2.1. Some facts about Legendre functions. Legendre functions mapping a general Banach space X into $(-\infty, +\infty]$ are defined in [**4**]. According to [**4**, Theorems 5.4 and 5.6], since X is reflexive, the function f is *Legendre* if and only if it satisfies the following two conditions:

(L1) The interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is Gâteaux differentiable (see below) on $\operatorname{int}\operatorname{dom} f$, and

$$\operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f;$$

(L2) The interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f^*$, and

$$\operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*.$$

Since X is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [**7**, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},$$

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f.$$

Also, conditions (L1) and (L2), in conjunction with [**4**, Theorem 5.4], imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Several interesting examples of Legendre functions are presented in [**3**] and [**4**]. Among them are the functions $\frac{1}{s} \|\cdot\|^s$ with $s \in (1, \infty)$, where the Banach space X is smooth and strictly convex and, in particular, a Hilbert space. From now on we assume that the convex function $f : X \to (-\infty, +\infty]$ is Legendre.

2.2. A Property of gradients. For any convex $f : X \to (-\infty, +\infty]$ we denote by $\operatorname{dom} f$ the set $\{x \in X : f(x) < +\infty\}$. For any $x \in \operatorname{int} \operatorname{dom} f$ and $y \in X$, we denote by $f^\circ(x, y)$ the *right-hand derivative of f at x* in the direction y, that is,

$$f^\circ(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called *Gâteaux differentiable* at x if $\lim_{t \to 0} (f(x + ty) - f(x))/t$ exists for any y. In this case $f^\circ(x, y)$ coincides with $(\nabla f)(x)$, the value of the *gradient* ∇f of f at x. The function f is said to be *Fréchet differentiable at x* if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable on a subset E of X* if the limit is attained uniformly for $x \in E$ and $\|y\| = 1$. We will need the following result.

Proposition 1 (*cf.* [**22**, Proposition 2.1, p. 474]). *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

2.3. Some facts about the Bregman distance. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$, defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \,, \tag{2.1}$$

is called the *Bregman distance with respect to* f (*cf.* [**14**]). The Bregman distance has the following two important properties, called the *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle \,, \tag{2.2}$$

and the *four point identity*: for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle \,. \tag{2.3}$$

2.4. Some facts about totally convex functions. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that, according to [**11**, Section 1.2, p. 17] (see also [**10**]), the function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if its *modulus of total convexity at* x, that is, the function $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x, t) := \inf \{ D_f(y, x) : y \in \operatorname{dom} f, \quad \|y - x\| = t \} \,,$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int}\operatorname{dom} f$. In addition, the function f is called *totally convex on bounded sets* if $\upsilon_f(E, t)$ is positive for any nonempty bounded subset E of X and for any $t > 0$, where the *modulus of total convexity of the function* f on the set E is the function $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\upsilon_f(E, t) := \inf \{ \upsilon_f(x, t) : x \in E \cap \operatorname{int}\operatorname{dom} f \} \,.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [**13**, Theorem 2.10, p. 9]).

Examples of totally convex functions can be found, for instance, in [**11, 13**].

The next proposition turns out to be very useful in the proof of Theorems 1 and 2 below.

Proposition 2 (*cf.* [**25**, Proposition 2.2, p. 3]). *If* $x \in \operatorname{int}\operatorname{dom} f$*, then the following statements are equivalent:*

(i) The function f *is totally convex at* x*;*

(ii) For any sequence $\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} f$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \implies \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Recall that the function f is called *sequentially consistent* (see [**13**]) if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Proposition 3 (*cf.* [**11**, Lemma 2.1.2, p. 67]). *The function* f *is totally convex on bounded sets if and only if it is sequentially consistent.*

Recall that the *Bregman projection* (*cf.* [**8**]) of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty, closed and convex set $K \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_K^f(x) \in K$ satisfying

$$D_f\left(\operatorname{proj}_K^f(x), x\right) = \inf\left\{D_f(y, x) : y \in K\right\}.$$

Similarly to the metric projection in Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Proposition 4 (*cf.* [**13**, Corollary 4.4, p. 23]). *Suppose that* f *is Gâteaux differentiable and totally convex on* $\operatorname{int}\operatorname{dom} f$*. Let* $x \in \operatorname{int}\operatorname{dom} f$ *and let* $K \subset \operatorname{int}\operatorname{dom} f$ *be a nonempty, closed and convex set. If* $\hat{x} \in K$*, then the following conditions are equivalent*:

(i) The vector $\hat{x}$ *is the Bregman projection of* x *onto* K *with respect to* f*;*

(ii) The vector $\hat{x}$ *is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \qquad \forall y \in K;$$

(iii) The vector $\hat{x}$ *is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \qquad \forall y \in K.$$

The following two propositions exhibit two additional properties of totally convex functions.

Proposition 5 (*cf.* [**23**, Lemma 3.1, p. 31]). *Let* $f : X \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in X$ *and the sequence* $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ *is bounded, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded too.*

Proposition 6 (*cf.* [**23**, Lemma 3.2, p. 31]). *Let* $f : X \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function,* $x_0 \in X$ *and let* K *be a nonempty, closed and convex subset of* X*. Suppose that the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded and any weak subsequential limit of* $\{x_n\}_{n\in\mathbb{N}}$ *belongs to* K*. If* $D_f(x_n, x_0) \leq D_f\left(\mathrm{proj}_K^f(x_0), x_0\right)$ *for any* $n \in \mathbb{N}$*, then* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_K^f(x_0)$.

2.5. Some facts about Bregman strongly nonexpansive operators. This class of operators was introduced in [**15**] and [**21**]. Let K be a convex subset of $\mathrm{int}\,\mathrm{dom}\, f$ and let T be a self-mapping of K. A point p in the closure of K is said to be an *asymptotic fixed point* of T (*cf.* [**15**] and [**21**]) if K contains a sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to p such that the strong $\lim_{n\to+\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. We say that the operator T is (*quasi-*) *Bregman strongly nonexpansive* (BSNE for short) with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.4}$$

for all $p \in \hat{F}(T)$ and $x \in K$, and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset K$ is bounded, $p \in \hat{F}(T)$, and

$$\lim_{n\to+\infty} \left(D_f(p, x_n) - D_f(p, Tx_n)\right) = 0, \tag{2.5}$$

it follows that

$$\lim_{n\to+\infty} D_f(Tx_n, x_n) = 0. \tag{2.6}$$

Note that the notion of a strongly nonexpansive operator (with respect to the norm) was first introduced and studied in [**9**] (see also [**20**]).

Another well-known family of operators is the class of Bregman firmly non-expansive operators, where an operator $T : K \to K$ is called *Bregman firmly nonexpansive (BFNE* for short*)* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \tag{2.7}$$

for all $x, y \in K$. It is clear from the definition of the Bregman distance (2.1) that inequality (2.7) is equivalent to

(2.8)

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

See [**5, 24**] for more information on BFNE operators. In particular, we prove in [**24**, Lemma 1.3.2] that for any BFNE operator T, $F(T) = \hat{F}(T)$ when the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X. In this case it also follows that any BFNE operator is a BSNE operator with respect to a nonempty $F(T) = \hat{F}(T)$.

Let $f : X \to \mathbb{R}$ be bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of K. In [**21**, Lemma 1, p. 314] it is shown that if $\{T_i : 1 \leq i \leq N\}$ are N BSNE operators on K and $\hat{F} = \bigcap\left\{\hat{F}(T_i) : 1 \leq i \leq N\right\}$ is not empty, then $\hat{F}(T_N T_{N-1} \cdots T_1)$ is contained in $\hat{F}$. Also, according to [**21**, Lemma 2, p. 314], if $\{T_i : 1 \leq i \leq N\}$ are BSNE operators, $T = T_N T_{N-1} \cdots T_1$, and the sets $\hat{F}(T)$ and $\hat{F}$ are nonempty, then T is BSNE too.

Let $\{T_i : 1 \leq i \leq N\}$ be N BSNE operators which satisfy $\hat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$ and let $T = T_N T_{N-1} \cdots T_1$. If $F = \bigcap \{F(T_i) : 1 \leq i \leq N\}$ and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \hat{F}(T)$. Indeed,

$$F(T) \subset \hat{F}(T) \subset \bigcap \left\{\hat{F}(T_i) : 1 \leq i \leq N\right\} = \bigcap \{F(T_i) : 1 \leq i \leq N\} \subset F(T),$$

which implies that $F(T) = \hat{F}(T)$, as claimed.

3. Two Strong Convergence Theorems

In this section we study the following algorithm when $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$:

$$\begin{cases} x_0 \in X, \\ y_n^i = T_i(x_n + e_n^i), \\ C_n^i = \left\{z \in X : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\}, \\ C_n := \cap_{i=1}^{N} C_n^i, \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n}(x_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{3.1}$$

Theorem 1. *Let $T_i : X \to X$, $i = 1, 2, \ldots, N$, be N BSNE operators which satisfy $F(T_i) = \hat{F}(T_i)$ for each $1 \leq i \leq N$ and $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}^f_F(x_0)$ as $n \to +\infty$.*

Proof. We begin with the following claim.

Claim 1. *There are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1).*

Let $n \in \mathbb{N}$. It is not difficult to check that the sets C_n^i are closed and convex for any $i = 1, 2, \ldots, N$. Hence their intersection C_n is also closed and convex. It is also obvious that Q_n is a closed and convex set. Let $u \in F$. For any $n \in \mathbb{N}$, we obtain from (2.4) that

$$D_f\left(u, y_n^i\right) = D_f\left(u, T_i(x_n + e_n^i)\right) \le D_f\left(u, x_n + e_n^i\right),$$

which implies that $u \in C_n^i$. Since this holds for any $i = 1, 2, \ldots, N$, it follows that $u \in C_n$. Thus $F \subset C_n$ for any $n \in \mathbb{N}$. On the other hand, it is obvious that $F \subset Q_0 = X$. Thus $F \subset C_0 \bigcap Q_0$, and therefore $x_1 = \operatorname{proj}^f_{C_0 \cap Q_0}(x_0)$ is well defined. Now suppose that $F \subset C_{n-1} \bigcap Q_{n-1}$ for some $n \ge 1$. Then $x_n = \operatorname{proj}^f_{C_{n-1} \cap Q_{n-1}}(x_0)$ is well defined because $C_{n-1} \bigcap Q_{n-1}$ is a nonempty, closed and convex subset of X. So from Proposition 4(ii) we have

$$\langle \nabla f(x_0) - \nabla f(x_n), y - x_n \rangle \le 0$$

for any $y \in C_{n-1} \bigcap Q_{n-1}$. Hence we obtain that $F \subset Q_n$. Therefore $F \subset C_n \bigcap Q_n$ and hence $x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n}(x_0)$ is also well defined. Consequently, we see that $F \subset C_n \bigcap Q_n$ for any $n \in \mathbb{N}$. Thus the sequence we constructed is indeed well defined and satisfies (3.1), as claimed.

From now on we fix an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying (3.1).

Claim 2. *The sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded.*

It follows from the definition of Q_n and Proposition 4(ii) that $\operatorname{proj}^f_{Q_n}(x_0) = x_n$. Furthermore, by Proposition 4(iii), for each $u \in F$, we have

$$\begin{aligned} D_f\left(x_n, x_0\right) &= D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \\ &\le D_f\left(u, x_0\right) - D_f\left(u, \operatorname{proj}^f_{Q_n}(x_0)\right) \\ &\le D_f\left(u, x_0\right). \end{aligned} \tag{3.2}$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in F$. Therefore by Proposition 5 the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded too, as claimed.

Claim 3. *Every weak subsequential limit of $\{x_n\}_{n \in \mathbb{N}}$ belongs to F.*

It follows from the definition of Q_n and Proposition 4(ii) that $\operatorname{proj}^f_{Q_n}(x_0) = x_n$. Since $x_{n+1} \in Q_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1}, \operatorname{proj}^f_{Q_n}(x_0)\right) + D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and hence

$$D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_0\right) \leq D_f\left(x_{n+1}, x_0\right). \tag{3.3}$$

Therefore the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is increasing and since it is also bounded (see Claim 2), $\lim_{n \to +\infty} D_f(x_n, x_0)$ exists. Thus it follows from (3.3) that

$$\lim_{n \to +\infty} D_f\left(x_{n+1}, x_n\right) = 0. \tag{3.4}$$

Proposition 3 now implies that

$$\lim_{n \to +\infty} \left(x_{n+1} - x_n\right) = 0. \tag{3.5}$$

For any $i = 1, 2, \ldots, N$, it follows from the definition of the Bregman distance (see (2.1)) that

$$D_f\left(x_n, x_n + e_n^i\right) = f\left(x_n\right) - f\left(x_n + e_n^i\right) - \left\langle \nabla f(x_n + e_n^i), x_n - \left(x_n + e_n^i\right)\right\rangle =$$
$$f\left(x_n\right) - f\left(x_n + e_n^i\right) + \left\langle \nabla f(x_n + e_n^i), e_n^i\right\rangle.$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [**11**, Proposition 1.1.11, p. 17]). In addition, f is uniformly Fréchet differentiable and therefore f is uniformly continuous on bounded subsets (see [**1**, Theorem 1.8, p. 13]). Hence, since $\lim_{n \to +\infty} e_n^i = 0$, we see that

$$\lim_{n \to +\infty} D_f\left(x_n, x_n + e_n^i\right) = 0. \tag{3.6}$$

For any $i = 1, 2, \ldots, N$, it follows from the three point identity (see (2.2)) that

$$D_f\left(x_{n+1}, x_n + e_n^i\right) = D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_n + e_n^i\right) + \left\langle \nabla f(x_n) - \nabla f(x_n + e_n^i), x_{n+1} - x_n \right\rangle.$$

Since ∇f is bounded on bounded subsets of X, (3.4), (3.5) and (3.6) imply that

$$\lim_{n \to +\infty} D_f\left(x_{n+1}, x_n + e_n^i\right) = 0.$$

Next, for any $i = 1, 2, \ldots, N$, it follows from the inclusion $x_{n+1} \in C_n^i$ that

$$D_f\left(x_{n+1}, y_n^i\right) \leq D_f\left(x_{n+1}, x_n + e_n^i\right).$$

Hence $\lim_{n \to +\infty} D_f\left(x_{n+1}, y_n^i\right) = 0$. Proposition 3 now implies that $\lim_{n \to +\infty}\left(y_n^i - x_{n+1}\right) = 0$. Therefore, for any $i = 1, 2, \ldots, N$, we have

$$\left\|y_n^i - x_n\right\| \leq \left\|y_n^i - x_{n+1}\right\| + \left\|x_{n+1} - x_n\right\| \to 0,$$

and since $\lim_{n \to +\infty} e_n^i = 0$, we obtain that

$$\lim_{n \to +\infty} \left\|y_n^i - (x_n + e_n^i)\right\| = 0. \tag{3.7}$$

This means that the sequence $\left\{y_n^i\right\}_{n \in \mathbb{N}}$ is bounded for any $i = 1, 2, \ldots, N$. Since f is uniformly Fréchet differentiable, it follows from Proposition 1 that

$$\lim_{n \to +\infty} \left\|\nabla f\left(y_n^i\right) - \nabla f\left(x_n\right)\right\|_* = 0$$

and since $\lim_{n \to +\infty} e_n^i = 0$, it also follows that

$$\lim_{n \to +\infty} \left\|\nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right)\right\|_* = 0 \tag{3.8}$$

for any $i = 1, 2, \ldots, N$. Since f is uniformly Fréchet differentiable, it is also uniformly continuous (see [**1**, Theorem 1.8, p. 13]) and therefore

$$\lim_{n \to +\infty} \left\|f\left(y_n^i\right) - f\left(x_n + e_n^i\right)\right\| = 0 \tag{3.9}$$

for any $i = 1, 2, \dots, N$. From the definition of the Bregman distance (see (2.1)) we obtain that

$$\begin{aligned}D_f\left(u, x_n + e_n^i\right) - D_f\left(u, y_n^i\right) &= \left[f(u) - f(x_n + e_n^i)\right. \\ &\quad \left.- \left\langle \nabla f(x_n + e_n^i), u - (x_n + e_n^i)\right\rangle\right] \\ &\quad - \left[f(u) - f(y_n^i) - \left\langle \nabla f(y_n^i), u - y_n^i\right\rangle\right] \\ &= f(y_n^i) - f(x_n + e_n^i) + \left\langle \nabla f(y_n^i), u - y_n^i\right\rangle \\ &\quad - \left\langle \nabla f(x_n + e_n^i), u - (x_n + e_n^i)\right\rangle \\ &= f(y_n^i) - f(x_n + e_n^i) + \left\langle \nabla f(y_n^i), x_n + e_n^i - y_n^i\right\rangle \\ &\quad + \left\langle \nabla f(y_n^i) - \nabla f(x_n + e_n^i), u - (x_n + e_n^i)\right\rangle\end{aligned}$$

for any $u \in F$. Since the sequence $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded, $\left\{\nabla f(y_n^i)\right\}_{n\in\mathbb{N}}$ is bounded too. Now from (3.7), (3.8) and (3.9), we obtain that

$$\lim_{n\to+\infty}\left(D_f\left(u, x_n + e_n^i\right) - D_f\left(u, y_n^i\right)\right) = 0$$

for any $u \in F$, that is,

$$\lim_{n\to+\infty}\left(D_f\left(u, x_n + e_n^i\right) - D_f\left(u, T_i(x_n + e_n^i)\right)\right) = 0$$

for any $u \in F$. Since each T_i, $1 \le i \le N$, is a BSNE operator which satisfies $F\left(T_i\right) = \hat{F}\left(T_i\right)$, it follows that

$$\lim_{n\to+\infty} D_f\left(T_i(x_n + e_n^i), x_n + e_n^i\right) = 0.$$

Proposition 3 now implies that

$$\lim_{n\to+\infty}\left(T_i(x_n + e_n^i) - (x_n + e_n^i)\right) = 0. \tag{3.10}$$

Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Set $z_n^i = x_n + e_n^i$. Since $x_{n_k} \rightharpoonup v$ and $e_{n_k}^i \to 0$, it is obvious that for any $1 \le i \le N$, the sequence $\left\{z_{n_k}^i\right\}_{k\in\mathbb{N}}$ converges weakly to v. We also have $\lim_{n\to+\infty}\left(T_i z^i_{n_k} - z^i_{n_k}\right) = 0$ by (3.10). This means that $v \in \hat{F}(T_i) = F(T_i)$. Therefore $v \in F$, as claimed. This proves Claim 3.

Claim 4. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^f_F(x_0)$ as $n \to +\infty$.*

Let $\tilde{u} = \mathrm{proj}^f_F(x_0)$. Since $x_{n+1} = \mathrm{proj}^f_{C_n\cap Q_n}(x_0)$ and F is contained in $C_n \bigcap Q_n$, we have $D_f(x_{n+1}, x_0) \le D_f(\tilde{u}, x_0)$. Therefore Proposition 6 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u} = \mathrm{proj}^f_F(x_0)$, as claimed.

This completes the proof of Theorem 1. □

We now present another result which is similar to Theorem 1, but with a different construction of the sequence $\{C_n\}_{n\in\mathbb{N}}$. The following algorithm is based on the concept of the so-called shrinking projection method which was introduced by Takahashi, Takeuchi and Kubota in [**27**]. More precisely, we study the following algorithm when $F := \bigcap_{i=1}^N F(T_i) \ne \varnothing$:

$$
\begin{cases}
x_0 \in X, \\
C^i_0 = X, \qquad i = 1, 2, \ldots, N, \\
y^i_n = T_i(x_n + e^i_n), \\
C^i_{n+1} = \left\{ z \in C^i_n : D_f\left(z, y^i_n\right) \le D_f\left(z, x_n + e^i_n\right) \right\}, \\
C_{n+1} := \cap_{i=1}^N C^i_{n+1}, \\
x_{n+1} = \mathrm{proj}^f_{C_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases}
\tag{3.11}
$$

Theorem 2. *Let $T_i : X \to X$, $i = 1, 2, \ldots, N$, be N BSNE operators which satisfy $\hat{F}(T_i) = F(T_i)$ for each $1 \le i \le N$ and $F := \bigcap_{i=1}^N F(T_i) \ne \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.11). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\left\{e^i_n\right\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e^i_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^f_F(x_0)$ as $n \to +\infty$.*

Proof. We begin with the following claim.

Claim 1. *There are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.11).*

Again it is not difficult to check that the sets C_n^i are closed and convex for any $i = 1, 2, \ldots, N$. Hence their intersection C_n is also closed and convex. Let $u \in F$. For any $n \in \mathbb{N}$, we obtain from (2.4) the inequality

$$D_f\left(u, y_n^i\right) = D_f\left(u, T_i(x_n + e_n^i)\right) \leq D_f\left(u, x_n + e_n^i\right),$$

which implies that $u \in C_{n+1}^i$. Since this holds for any $i = 1, 2, \ldots, N$, it follows that $u \in C_{n+1}$. Thus $F \subset C_n$ for any $n \in \mathbb{N}$.

From now on we fix an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (3.11).

Claim 2. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.*

It follows from Proposition 4(iii) that, for each $u \in F$, we have

$$\begin{aligned} D_f\left(x_n, x_0\right) &= D_f\left(\operatorname{proj}_{C_n}^f(x_0), x_0\right) \\ &\leq D_f\left(u, x_0\right) - D_f\left(u, \operatorname{proj}_{C_n}^f(x_0)\right) \\ &\leq D_f\left(u, x_0\right). \end{aligned} \tag{3.12}$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in F$. Therefore by Proposition 5 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Claim 3. *Every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to F.*

Since $x_{n+1} \in C_{n+1} \subset C_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1}, \operatorname{proj}_{C_n}^f(x_0)\right) + D_f\left(\operatorname{proj}_{C_n}^f(x_0), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and hence

$$D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_0\right) \leq D_f\left(x_{n+1}, x_0\right). \tag{3.13}$$

Therefore the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Claim 2), $\lim_{n\to+\infty} D_f(x_n, x_0)$ exists. Thus it follows from (3.13) that

$$\lim_{n\to+\infty} D_f\left(x_{n+1}, x_n\right) = 0. \tag{3.14}$$

Now, using an argument similar to the one we employed in the proof of Theorem 1 (see Claim 3 there), we get the conclusion of Claim 3.

Claim 4. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_F^f(x_0)$ *as* $n \to +\infty$.

Let $\tilde{u} = \mathrm{proj}_F^f(x_0)$. Since $x_n = \mathrm{proj}_{C_n}^f(x_0)$ and F is contained in C_n, we have $D_f(x_n, x_0) \leq D_f(\tilde{u}, x_0)$. Therefore Proposition 6 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u} = \mathrm{proj}_F^f(x_0)$, as claimed.

This completes the proof of Theorem 2. $\square$

4. Convex Feasibility Problems

Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. The convex feasibility problem (CFP) is to find an element in the assumed nonempty intersection $\bigcap_{i=1}^N K_i$. It is clear that $F\left(\mathrm{proj}_{K_i}^f\right) = K_i$ for any $i = 1, 2, \ldots, N$. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the Bregman projection $\mathrm{proj}_{K_i}^f$ is BFNE, hence BSNE, and $F\left(\mathrm{proj}_{K_i}^f\right) = \hat{F}\left(\mathrm{proj}_{K_i}^f\right)$ (*cf.* [**24**, Lemma 1.3.2]). Therefore, if we take $T_i = \mathrm{proj}_{K_i}^f$ in Theorems 1 and 2, then we get two different algorithms for solving convex feasibility problems which allow for computational errors.

Corollary 1. *Let* K_i*,* $i = 1, 2, \ldots, N$*, be* N *nonempty, closed and convex subsets of* X *such that* $K := \bigcap_{i=1}^N K_i \neq \varnothing$*. Let* $f : X \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X*. Then, for each* $x_0 \in X$*, there are sequences* $\{x_n\}_{n\in\mathbb{N}}$ *which satisfy (3.1) (with* $T_i = \mathrm{proj}_{K_i}^f$*). If the sequences of errors* $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ *satisfy* $\lim_{n\to+\infty} e_n^i = 0$*, then each such sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_K^f(x_0)$ *as* $n \to +\infty$.

Corollary 2. *Let* K_i*,* $i = 1, 2, \ldots, N$*, be* N *nonempty, closed and convex subsets of* X *such that* $K := \bigcap_{i=1}^N K_i \neq \varnothing$*. Let* $f : X \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X*. Then, for each* $x_0 \in X$*, there are sequences* $\{x_n\}_{n\in\mathbb{N}}$ *which satisfy (3.11) (with* $T_i = \mathrm{proj}_{K_i}^f$*). If the sequences of errors* $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ *satisfy*

$\lim_{n\to+\infty} e_n^i = 0$, *then each such sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_K^f(x_0)$ *as* $n \to +\infty$.

5. Zeroes of Maximal Monotone Operators

Let $A : X \to 2^{X^*}$ be a maximal monotone operator. The problem of finding an element $x \in X$ such that $0^* \in Ax$ is very important in Optimization Theory and related fields. In this section we present two different algorithms for finding common zeroes of N maximal monotone operators.

Recall that the resolvent of A, denoted by $\mathrm{Res}_A^f : X \to 2^X$, is defined as follows [**5**]:

$$\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

Bauschke, Borwein and Combettes [**5**, Prop. 3.8(iv), p. 604] prove that this resolvent is a single-valued BFNE operator. In addition, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the resolvent Res_A^f is a BSNE operator (see Section 2.5) which satisfies $F\left(\mathrm{Res}_A^f\right) = \hat{F}\left(\mathrm{Res}_A^f\right)$ (*cf.* [**24**, Lemma 1.3.2]). It is well known that the fixed point set of the resolvent Res_A^f is equal to the set of zeroes of the operator A, that is, $F\left(\mathrm{Res}_A^f\right) = A^{-1}(0^*)$. If we take $T_i = \mathrm{Res}_{A_i}^f$ in Theorems 1 and 2, then we obtain two different algorithms for finding common zeroes of finitely many maximal monotone operators which allow for computational errors. Note that since each A_i is a maximal monotone operator, $X^* = \mathrm{ran}\,(\nabla f) = \mathrm{ran}\,(\nabla f + A_i)$ (see [**23**, Proposition 2.3, p. 28] and [**5**, Prop. 3.8(iv), p. 604]) and therefore each T_i is defined on all of X.

Corollary 3. *Let* $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, *be* N *maximal monotone operators with* $Z := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \varnothing$. *Let* $f : X \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X. *Then, for each* $x_0 \in X$, *there are sequences* $\{x_n\}_{n\in\mathbb{N}}$ *which satisfy (3.1) (with* $T_i = \mathrm{Res}_{A_i}^f$*). If, for each* $i = 1, 2, \ldots, N$, *the sequences of errors*

$\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ *satisfy* $\lim_{n\to+\infty} e_n^i = 0$*, then each such sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *converges strongly to* $\operatorname{proj}_Z^f(x_0)$ *as* $n \to +\infty$.

Corollary 4. *Let* $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, *be* N *maximal monotone operators with* $Z := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \varnothing$. *Let* $f : X \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X. *Then, for each* $x_0 \in X$, *there are sequences* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *which satisfy (3.11) (with* $T_i = \operatorname{Res}_{A_i}^f$*). If, for each* $i = 1, 2, \ldots, N$, *the sequences of errors* $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ *satisfy* $\lim_{n\to+\infty} e_n^i = 0$*, then each such sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *converges strongly to* $\operatorname{proj}_Z^f(x_0)$ *as* $n \to +\infty$.

6. Equilibrium Problems

Let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies the following conditions [**6**]:

(C1) $g(x, x) = 0$ for all $x \in K$;

(C2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;

(C3) for all $x, y, z \in K$,

$$\limsup_{t\downarrow 0} g\left(tz + (1-t)x, y\right) \leq g(x, y);$$

(C4) for each $x \in K$, $g(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problem corresponding to g is to find $\bar{x} \in K$ such that

$$g(\bar{x}, y) \geq 0 \quad \forall y \in K. \tag{6.1}$$

The set of solutions of (6.1) is denoted by $EP(g)$.

The *resolvent of a bifunction* $g : K \times K \to \mathbb{R}$ [**16**] is the operator $\operatorname{Res}_g^f : X \to 2^K$, defined by

$$\operatorname{Res}_g^f(x) = \left\{z \in K : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \quad \forall y \in K\right\}.$$

In the following two lemmata we obtain several properties of these resolvents. We first show that $\operatorname{dom}(\mathrm{Res}_g^f)$ is the whole space X when f is a *coercive* (*i.e.*, $\lim_{\|x\|\to+\infty}(f(x)/\|x\|)=+\infty$) and Gâteaux differentiable function.

Lemma 1. *Let $f:X\to(-\infty,+\infty]$ be a coercive and Gâteaux differentiable function. Let K be a closed and convex subset of X. If the bifunction $g:K\times K\to\mathbb{R}$ satisfies conditions* (C1)–(C4), *then* $\operatorname{dom}(\mathrm{Res}_g^f)=X$.

Proof. First we show that for any $\xi\in X^*$, there exists $\bar{x}\in K$ such that

$$g(\bar{x},y)+f(y)-f(\bar{x})-\langle\xi,y-\bar{x}\rangle\geq 0 \tag{6.2}$$

for any $y\in K$. Since f is a coercive function, the function $h:X\times X\to(-\infty,+\infty]$, defined by

$$h(x,y)=f(y)-f(x)-\langle\xi,y-x\rangle,$$

satisfies

$$\lim_{\|x-y\|\to+\infty}\frac{h(x,y)}{\|x-y\|}=-\infty$$

for each fixed $y\in K$. Therefore it follows from Theorem 1 in [**6**] that (6.2) holds. Now we prove that (6.2) implies that

$$g(\bar{x},y)+\langle\nabla f(\bar{x}),y-\bar{x}\rangle-\langle\xi,y-\bar{x}\rangle\geq 0$$

for any $y\in K$. We know that (6.2) holds for $y=t\bar{x}+(1-t)\bar{y}$, where $\bar{y}\in K$ and $t\in(0,1)$. Hence,

$$g(\bar{x},t\bar{x}+(1-t)\bar{y})+f(t\bar{x}+(1-t)\bar{y})-f(\bar{x})-\langle\xi,t\bar{x}+(1-t)\bar{y}-\bar{x}\rangle\geq 0 \tag{6.3}$$

for all $\bar{y}\in K$. Since

$$f(t\bar{x}+(1-t)\bar{y})-f(\bar{x})\leq\langle\nabla f(t\bar{x}+(1-t)\bar{y}),t\bar{x}+(1-t)\bar{y}-\bar{x}\rangle,$$

we get from (6.3) and condition (C4) that

$$tg\left(\bar{x},\bar{x}\right)+\left(1-t\right)g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(t\bar{x}+\left(1-t\right)\bar{y}\right),t\bar{x}+\left(1-t\right)\bar{y}-\bar{x}\right\rangle$$
$$-\left\langle \xi,t\bar{x}+\left(1-t\right)\bar{y}-\bar{x}\right\rangle \geq 0$$

for all $\bar{y}\in K$. From condition (C1) we know that $g\left(\bar{x},\bar{x}\right)=0$. So, we have

$$\left(1-t\right)g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(t\bar{x}+\left(1-t\right)\bar{y}\right),\left(1-t\right)\left(\bar{y}-\bar{x}\right)\right\rangle -\left\langle \xi,\left(1-t\right)\left(\bar{y}-\bar{x}\right)\right\rangle \geq 0$$

and

$$\left(1-t\right)\left[g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(t\bar{x}+\left(1-t\right)\bar{y}\right),\bar{y}-\bar{x}\right\rangle -\left\langle \xi,\bar{y}-\bar{x}\right\rangle\right] \geq 0$$

for all $\bar{y}\in K$. Therefore

$$g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(t\bar{x}+\left(1-t\right)\bar{y}\right),\bar{y}-\bar{x}\right\rangle -\left\langle \xi,\bar{y}-\bar{x}\right\rangle \geq 0$$

for all $\bar{y}\in K$. Since f is a Gâteaux differentiable function, it follows that ∇f is norm-to-weak* continuous (see [**19**, Propostion 2.8, p. 19]). Therefore, letting here $t\to 1^{-}$, we obtain that

$$g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(\bar{x}\right),\bar{y}-\bar{x}\right\rangle -\left\langle \xi,\bar{y}-\bar{x}\right\rangle \geq 0$$

for all $\bar{y}\in K$. Hence, for any $x\in X$, taking $\xi=\nabla f\left(x\right)$, we obtain $\bar{x}\in K$ such that

$$g\left(\bar{x},\bar{y}\right)+\left\langle \nabla f\left(\bar{x}\right)-\nabla f\left(x\right),\bar{y}-\bar{x}\right\rangle \geq 0$$

for all $\bar{y}\in K$, that is, $\bar{x}\in \mathrm{Res}_g^f\left(x\right)$. Hence $\mathrm{dom}\left(\mathrm{Res}_g^f\right)=X$. □

In the next lemma we list more properties of the resolvent of a bifunction.

Lemma 2. *Let $f:X\to(-\infty,+\infty]$ be a Legendre function. Let K be a closed and convex subset of X. If the bifunction $g:K\times K\to\mathbb{R}$ satisfies conditions* (C1)–(C4), *then*

(i) Res_g^f *is single-valued;*

(ii) Res_g^f *is a BFNE operator;*

(iii) the set of fixed points of Res_g^f *is the solution set of the corresponding equilibrium problem, i.e.,* $F(\operatorname{Res}_g^f) = EP\,(g)$;

(iv) $EP\,(g)$ *is a closed and convex subset of* K;

(v) For all $x \in X$ *and for all* $u \in F(\operatorname{Res}_g^f)$, *we have*

$$D_f\left(u, \operatorname{Res}_g^f(x)\right) + D_f\left(\operatorname{Res}_g^f(x), x\right) \le D_f(u, x).$$

Proof. *(i)* Let $z_1, z_2 \in \operatorname{Res}_g^f(x)$. Then the definition of the resolvent implies that

$$g(z_1, z_2) + \langle \nabla f(z_1) - \nabla f(x), z_2 - z_1 \rangle \ge 0$$

and

$$g(z_2, z_1) + \langle \nabla f(z_2) - \nabla f(x), z_1 - z_2 \rangle \ge 0.$$

Adding these two inequalities, we obtain

$$g(z_1, z_2) + g(z_2, z_1) + \langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2 \rangle \ge 0.$$

From condition (C2) it follows that

$$\langle \nabla f(z_2) - \nabla f(z_1), z_1 - z_2 \rangle \ge 0.$$

The function f is Legendre and therefore it is strictly convex. Hence ∇f is strictly monotone and therefore $z_1 = z_2$.

(ii) For any $x, y \in K$, we have

$$g\left(\operatorname{Res}_g^f(x), \operatorname{Res}_g^f(y)\right) + \left\langle \nabla f\left(\operatorname{Res}_g^f(x)\right) - \nabla f(x), \operatorname{Res}_g^f(y) - \operatorname{Res}_g^f(x) \right\rangle \ge 0$$

and

$$g\left(\operatorname{Res}_g^f(y), \operatorname{Res}_g^f(x)\right) + \left\langle \nabla f\left(\operatorname{Res}_g^f(y)\right) - \nabla f(y), \operatorname{Res}_g^f(x) - \operatorname{Res}_g^f(y) \right\rangle \ge 0.$$

Adding these two inequalities, we obtain that

$$g\left(\mathrm{Res}_g^f(x), \mathrm{Res}_g^f(y)\right) + g\left(\mathrm{Res}_g^f(y), \mathrm{Res}_g^f(x)\right)$$
$$+\left\langle \nabla f\left(\mathrm{Res}_g^f(x)\right) - \nabla f(x) + \nabla f(y) - \nabla f\left(\mathrm{Res}_g^f(y)\right), \mathrm{Res}_g^f(y) - \mathrm{Res}_g^f(x)\right\rangle \geq 0.$$

From condition (C2) it follows that

$$\left\langle \nabla f\left(\mathrm{Res}_g^f(x)\right) - \nabla f(x) + \nabla f(y) - \nabla f\left(\mathrm{Res}_g^f(y)\right), \mathrm{Res}_g^f(y) - \mathrm{Res}_g^f(x)\right\rangle \geq 0.$$

Hence

$$\begin{aligned}&\left\langle \nabla f\left(\mathrm{Res}_g^f(x)\right) - \nabla f\left(\mathrm{Res}_g^f(y)\right), \mathrm{Res}_g^f(x) - \mathrm{Res}_g^f(y)\right\rangle \\ &\leq \left\langle \nabla f(x) - \nabla f(y), \mathrm{Res}_g^f(x) - \mathrm{Res}_g^f(y)\right\rangle.\end{aligned}$$

This means that Res_g^f is a BFNE operator, as claimed.

(iii) Indeed,

$$\begin{aligned}x \in F(\mathrm{Res}_g^f) &\Longleftrightarrow x = \mathrm{Res}_g^f(x) \\ &\Longleftrightarrow 0 \leq g(x,y) + \langle \nabla f(x) - \nabla f(x), y - x\rangle \quad \forall y \in K \\ &\Longleftrightarrow 0 \leq g(x,y) \quad \forall y \in K \\ &\Longleftrightarrow x \in EP(g).\end{aligned}$$

(iv) Since Res_g^f is a BFNE operator, it follows from [**24**, Lemma 1.3.1] that $F(\mathrm{Res}_g^f)$ is a closed and convex subset of C. Therefore from *(iii)* we obtain that $EP(g) = F(\mathrm{Res}_g^f)$ is also a closed and convex subset of K, as claimed.

(v) Since Res_g^f is a BFNE operator, it follows from (3.11) that for all $x, y \in X$, we have

$$\begin{aligned}&D_f\left(\mathrm{Res}_g^f(x), \mathrm{Res}_g^f(y)\right) + D_f\left(\mathrm{Res}_g^f(y), \mathrm{Res}_g^f(x)\right) \\ &\leq D_f\left(\mathrm{Res}_g^f(x), y\right) - D_f\left(\mathrm{Res}_g^f(x), x\right) \\ &+ D_f\left(\mathrm{Res}_g^f(y), x\right) - D_f\left(\mathrm{Res}_g^f(y), y\right).\end{aligned}$$

Letting $y = u \in F(\mathrm{Res}_g^f)$, we see that

$$D_f\left(\mathrm{Res}_g^f(x), u\right) + D_f\left(u, \mathrm{Res}_g^f(x)\right) \le D_f\left(\mathrm{Res}_g^f(x), u\right) - D_f\left(\mathrm{Res}_g^f(x), x\right) + D_f(u, x) - D_f(u, u).$$

Thus

$$D_f\left(u, \mathrm{Res}_g^f(x)\right) + D_f\left(\mathrm{Res}_g^f(x), x\right) \le D_f(u, x).$$

This completes the proof. □

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the resolvent Res_g^f is single-valued (Lemma 2(i)), BSNE (see Section 2.5) and satisfies $F\left(\mathrm{Res}_g^f\right) = \hat{F}\left(\mathrm{Res}_g^f\right)$ (*cf.* [**24**, Lemma 1.3.2]). From Lemma 2(iii) we also know that $F\left(\mathrm{Res}_g^f\right) = EP(g)$. So, if we take $T_i = \mathrm{Res}_{g_i}^f$ in Theorems 1 and 2, then we get two different algorithms for finding common solutions to the equilibrium problems corresponding to finitely many bifunctions, which allow for computational errors. Note that each T_i is defined on all of X (Lemma 1).

Corollary 5. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) such that $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (3.1) (with $T_i = \mathrm{Res}_{g_i}^f$). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\left\{e_n^i\right\}_{n \in \mathbb{N}} \subset X$ satisfy $\lim_{n \to +\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $\mathrm{proj}_E^f(x_0)$ as $n \to +\infty$.*

Corollary 6. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) such that $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences*

$\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.11) (with $T_i = \mathrm{Res}^f_{g_i}$). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\{e^i_n\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e^i_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^f_E(x_0)$ as $n \to +\infty$.

7. Zeroes of Bregman Inverse Strongly Monotone Operators

Using our methods, we can find common zeroes for another class of operators, namely, Bregman inverse strongly monotone operators. This class of operators was introduced by Butnariu and Kassay (see [**12**]). We assume that the Legendre function f satisfies the following range condition:

$$\mathrm{ran}\,(\nabla f - A) \subseteq \mathrm{ran}\,(\nabla f)\,. \tag{7.1}$$

The operator $A : X \to 2^{X^*}$ is called *Bregman inverse strongly monotone* (BISM for short) if

$$(\mathrm{dom}\,A) \bigcap (\mathrm{int}\,\mathrm{dom} f) \neq \varnothing$$

and for any $x, y \in \mathrm{int}\,\mathrm{dom} f$, and each $\xi \in Ax$, $\eta \in Ay$, we have

$$\langle \xi - \eta, \nabla f^*\left(\nabla f\left(x\right) - \xi\right) - \nabla f^*\left(\nabla f\left(y\right) - \eta\right)\rangle \geq 0.$$

For any operator $A : X \to 2^{X^*}$, the *anti-resolvent* $A^f : X \to 2^X$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A)\,.$$

Observe that $\mathrm{dom}\,A^f \subseteq (\mathrm{dom}\,A) \cap (\mathrm{int}\,\mathrm{dom} f)$ and $\mathrm{ran}\,A^f \subseteq \mathrm{int}\,\mathrm{dom} f$.

It is known (see [**12**, Lemma 3.5 (c) and (d), p. 2109]) that the operator A is BISM if and only if the anti-resolvent A^f is a (single-valued) BFNE operator. For examples of BISM operators and more information on this new class of operators see [**12**]. Before presenting consequences of our main results, we note several properties of this classs of operators and of the anti-resolvent.

From the definition of the anti-resolvent and [**12**, Lemma 3.5, p. 2109] we obtain the following proposition.

Proposition 7. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function and let $A : X \to 2^{X^*}$ be a BISM operator such that $A^{-1}(0^*) \neq \varnothing$. Then the following statements hold:*

(i) $A^{-1}(0^*) = F(A^f)$;

(ii) For any $u \in A^{-1}(0^*)$ and $x \in \operatorname{dom} A^f$, we have

$$D_f\left(u, A^f x\right) + D_f\left(A^f x, x\right) \leq D_f\left(u, x\right).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the anti-resolvent A^f is a single-valued BSNE operator (see Section 2.5) which satisfies $F\left(A^f\right) = \hat{F}\left(A^f\right)$ (*cf.* [**24**, Lemma 1.3.2]).

Let K_i, $i = 1, 2, \ldots, N$, *be* N nonempty, closed and convex subsets of X and let $T_i : K_i \to K_i$ for each $i = 1, 2, \ldots, N$. Assume that $\bigcap_{i=1}^{N} K_i \neq \varnothing$ and consider Algorithms (3.1) and (3.11) without computational errors:

$$\tag{7.2} \begin{cases} x_0 \in K = \bigcap_{i=1}^{N} K_i, \\ y_n^i = T_i(x_n), \\ C_n^i = \left\{ z \in K_i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n\right) \right\}, \\ C_n := \bigcap_{i=1}^{N} C_n^i, \\ Q_n = \left\{ z \in K : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0 \right\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f (x_0), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

and

$$\tag{7.3} \begin{cases} x_0 \in K = \bigcap_{i=1}^{N} K_i, \\ C_0^i = K_i, \qquad i = 1, 2, \ldots, N, \\ y_n^i = T_i(x_n), \\ C_{n+1}^i = \left\{ z \in C_n^i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n\right) \right\}, \\ C_{n+1} := \bigcap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = \operatorname{proj}_{C_{n+1}}^f (x_0), \qquad n = 0, 1, 2, \ldots. \end{cases}$$

In the next two results we assume that each one of the operators A_i satisfies $K_i \subset \operatorname{dom} A_i$ and that $f : X \to \mathbb{R}$. From the range condition (7.1) we get that $\operatorname{dom} A_i^f = (\operatorname{dom} A_i) \cap (\operatorname{int} \operatorname{dom} f) = \operatorname{dom} A_i$ because in our case $\operatorname{int} \operatorname{dom} f = X$. From Proposition 7(i) we know that $F\left(A_i^f\right) = A_i^{-1}(0^*)$. So, if we take $T_i = A_i^f$ in Theorems 1 and 2, then we get two different algorithms for finding common zeroes of finitely many BISM operators.

Corollary 7. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N BISM operators such that $K_i \subset \operatorname{dom} A_i$ and $Z := \bigcap_{i=1}^{N} A_i^{-1}(0^*) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that the range condition (7.1) is satisfied for each A_i. Then, for each $x_0 \in K$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (7.2) (with $T_i = A_i^f$) and each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_Z^f(x_0)$ as $n \to +\infty$.*

Corollary 8. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N BISM operators such that $K_i \subset \operatorname{dom} A_i$ and $Z := \bigcap_{i=1}^{N} A_i^{-1}(0^*) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that the range condition (7.1) is satisfied for each A_i. Then, for each $x_0 \in K$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (7.3) (with $T_i = A_i^f$) and each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_Z^f(x_0)$ as $n \to +\infty$.*

8. Variational Inequalities

Let $A : X \to X^*$ be a BISM operator and let K be a nonempty, closed and convex subset of $\operatorname{dom} A$. The variational inequality problem corresponding to A is to find $\bar{x} \in K$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall y \in K. \tag{8.1}$$

The set of solutions of (8.1) is denoted by $VI(A,K)$.

In the following result we point out the connection between the fixed point set of $\mathrm{proj}_K^f \circ A^f$ and the solution set of the variational inequality corresponding to the BISM operator A.

Proposition 8. *Let $f : X \to (-\infty, +\infty]$ be a Legendre and totally convex function which satisfies the range condition (7.1). Let $A : X \to X^*$ be a BISM operator. If K is a nonempty, closed and convex subset of $\operatorname{dom} A \bigcap \operatorname{int}\operatorname{dom} f$, then $VI(K,A) = F(\mathrm{proj}_K^f \circ A^f)$.*

Proof. From Proposition 4 (ii) we obtain that $\bar{x} = \mathrm{proj}_K^f \left(A^f \bar{x}\right)$ if and only if

$$\left\langle \nabla f\left(A^f \bar{x}\right) - \nabla f\left(\bar{x}\right), \bar{x} - y \right\rangle \geq 0$$

for all $y \in K$, and this is equivalent to

$$\langle (\nabla f\left(\bar{x}\right) - A\bar{x}) - \nabla f\left(\bar{x}\right), \bar{x} - y \rangle \geq 0$$

for each $y \in K$, that is,

$$\langle -A\bar{x}, \bar{x} - y \rangle \geq 0$$

for any $y \in K$, which is equivalent to $\bar{x} \in VI(K,A)$, as claimed. □

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the anti-resolvent A^f is a single-valued [**12**, Lemma 3.5(d), p.2109] BSNE operator (see Section 2.5 and [**12**, Lemma 3.5(c), p.2109]) which satisfies $F\left(A^f\right) = \hat{F}\left(A^f\right)$ (*cf.* [**24**, Lemma 1.3.2]). Since the Bregman projection proj_K^f is a BFNE operator, it is also a BSNE operator (see Section 2.5) which satisfies $F\left(\mathrm{proj}_K^f\right) = \hat{F}\left(\mathrm{proj}_k^f\right)$. It now follows from [**21**, Lemma 2, p. 314] that $\mathrm{proj}_K^f \circ A^f$ is also a BSNE operator which satisfies $F\left(\mathrm{proj}_K^f \circ A^f\right) = \hat{F}\left(\mathrm{proj}_K^f \circ A^f\right)$ (see also Section 2.5 for more details). From Proposition 8 we know that $F\left(\mathrm{proj}_K^f \circ A^f\right) = VI(K,A)$. In this case we also employ Algorithms (7.2) and (7.3). Hence, if we take $T_i = \mathrm{proj}_{K_i}^f \circ A_i^f$ in Theorems 1 and 2, then we get two different algorithms for finding a solution to the (common) variational inequality problem corresponding to finitely many BISM operators.

Corollary 9. *Let K_i, $i = 1, 2, \dots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to X^*$, $i = 1, 2, \dots, N$, be N BISM operators such that $K_i \subset \operatorname{dom} A_i$ and $V := \bigcap_{i=1}^{N} VI(K_i, A_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that the range condition (7.1) is satisfied for each A_i. Then, for each $x_0 \in K$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (7.2) (with $T_i = \operatorname{proj}_{K_i}^f \circ A_i^f$) and each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $\operatorname{proj}_V^f(x_0)$ as $n \to +\infty$.*

Corollary 10. *Let K_i, $i = 1, 2, \dots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to X^*$, $i = 1, 2, \dots, N$, be N BISM operators such that $K_i \subset \operatorname{dom} A_i$ and $V := \bigcap_{i=1}^{N} VI(K_i, A_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that the range condition (7.1) is satisfied for each A_i. Then, for each $x_0 \in K$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (7.3) (with $T_i = \operatorname{proj}_{K_i}^f \circ A_i^f$) and each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $\operatorname{proj}_V^f(x_0)$ as $n \to +\infty$.*

Remark 1. In both Sections 7 and 8 we can still allow for possible computational errors if we assume that there exists $\varepsilon > 0$ such that $\left\| e_n^i \right\| < \varepsilon$ for each $i = 1, 2, \dots, N$ and for all $n \in \mathbb{N}$, and that the relevant operators are defined not only on K, but also on $K_\varepsilon := \{x \in X : d(x, K) < \varepsilon\}$, where $d(x, K) := \inf \{\|x - y\| : y \in K\}$.

9. Mixed Problems

There are many papers which propose algorithms for finding common solutions to mixed problems, for example, common solutions to two fixed point problems and, say, an equilibrium problem. If we combine Sections 4–6, then we can find common solutions to any finite number of problems such as fixed point problems, CFP, finding zeroes of maximal monotone operators and EP.

For instance, if we wish to find a common solution to two fixed point problems and an equilibrium problem, then we define the operators in Theorems 1 and 2 as follows: $T_1 = T$, $T_2 = S$ and $T_3 = \mathrm{Res}_g^f$, where T and S are BSNE operators which satisfy $F(T_i) = \hat{F}(T_i)$, $i = 1, 2$, and $g : C \times C \to \mathbb{R}$ is a bifunction that satisfies conditions (C1)–(C4). If

$$F = F(T) \bigcap F(S) \bigcap EP(g) \neq \varnothing,$$

then Algorithms (3.1) and (3.11) generate sequences $\{x_n\}_{n \in \mathbb{N}}$ which converge strongly to $\mathrm{proj}_F^f(x_0)$ as $n \to +\infty$.

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