Existence and Approximation of Fixed Points of Bregman Firmly Nonexpansive Mappings in Reflexive Banach Spaces

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Abstract. We study the existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces.

1. Introduction

In this paper X denotes a real reflexive Banach space with norm $\|\cdot\|$ and X^* stands for the (topological) dual of X endowed with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x \rangle$. An operator $A : X \to 2^{X^*}$ is said to be *monotone* if for any $x, y \in \operatorname{dom} A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \quad \Longrightarrow \quad \langle \xi - \eta, x - y \rangle \geq 0.$$

(Recall that the set $\operatorname{dom} A = \{x \in X \ : \ Ax \neq \varnothing\}$ is called the *effective domain* of such an operator A.) A monotone operator A is said to be *maximal* if graph A, the graph of A, is not a proper subset of the graph of any other monotone operator. In this paper $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and convex function, and $f^* : X^* \to (-\infty, +\infty]$ is the Fenchel conjugate of f. A *sublevel set* of f is a set of the form $\operatorname{lev}_{\leq}^{f}(r) = \{x \in X \ : \ f(x) \leq r\}$ for some $r \in \mathbb{R}$. We say that f is *positively homogeneous of degree* $\alpha \in \mathbb{R}$ if $f(tx) = t^{\alpha} f(x)$ for all $x \in X$ and $t > 0$. The set of nonnegative integers will be denoted by $\mathbb{N}$.

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Let C be a nonempty, closed and convex subset of a Hilbert space H. Then a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It turns out that nonexpansive fixed point theory can be applied to the problem of finding a point $z \in H$ satisfying

$$0 \in Az,$$

where $A : H \to 2^H$ is a maximal monotone operator. A key tool for solving this problem is the classical *resolvent* of A which is defined by $R_A = (I + A)^{-1}$. This resolvent is not only nonexpansive but also a *firmly nonexpansive* mapping, that is,

$$\|R_A x - R_A y\|^2 \leq \langle R_A x - R_A y, x - y\rangle$$

for all $x, y \in H$ (the resolvent R_A has full domain H when A is maximal monotone). See [**22**], [**11**] and [**17**] for more details. We also have $F(R_A) = A^{-1}(0)$, where $F(R_A)$ stands for the set of fixed points of R_A. Thus the problem of finding zeroes of maximal monotone operators in Hilbert space is reduced to that of finding fixed points of firmly nonexpansive mappings. In particular, if A is the subdifferential ∂f of f, then R_A is given by

$$R_A x = \operatorname{argmin}_{y \in H} \left\{ f(y) + \frac{1}{2} \|y - x\|^2 \right\}$$

for all $x \in H$ [**23**]. In this case, $F(R_A) = \{z \in H \mid f(z) = \inf_{y \in H} f(y)\}$.

The notion of a firmly nonexpansive mapping was extended to Banach spaces in [**10**] and [**11**]; see also [**17**]. However, in contrast with the case of Hilbert space, the resolvent of a maximal monotone operator is not, in general, even a nonexpansive mapping in the case of Banach spaces. Many other types of resolvents have been studied. For example, Alber [**1**], and Kohsaka and Takahashi [**19, 20, 21**] initiated the study of a generalized resolvent based on the duality mapping J.

Recently, Kohsaka and Takahashi [**20, 21**] have introduced the class of mappings of *firmly nonexpansive type*. Such a mapping T satisfies

$$\langle JTx - JTy, Tx - Ty\rangle \leq \langle Jx - Jy, Tx - Ty\rangle$$

for all $x, y \in C$, where J is the duality mapping of the Banach space X, and C is a nonempty, closed and convex subset of X. It is obvious that if we return to Hilbert space, then $J = I$ and the definitions of a firmly nonexpansive mapping and a mapping of firmly nonexpansive type coincide. Kohsaka and Takahashi prove that the generalized resolvent is a mapping of firmly nonexpansive type when X is a smooth, strictly convex and reflexive Banach space.

Even earlier, Bauschke, Borwein and Combettes [**4**] generalized the class of firmly nonexpansive mappings on smooth, strictly convex and reflexive Banach spaces to the case of general reflexive Banach spaces. Their mappings do not depend on the duality mapping J, but on the gradient ∇f of a well chosen function f. They call those mappings D_f-firmly nonexpansive mappings. In this paper we call them Bregman firmly nonexpansive mappings (BFNE in short) with respect to the function f. Bauschke, Borwein and Combettes prove that the resolvent based on the gradient ∇f of a well chosen function f is a BFNE mapping.

Our aim in this paper is to study the existence and approximation of fixed points of BFNE mappings in reflexive Banach spaces. In Section 2 we present several preliminary definitions and results. The third section is devoted to two properties of BFNE mappings. In the fourth section we prove two existence theorems (Theorems 1 and 2) regarding fixed points of a single BFNE mappings, as well as a common fixed point theorem (Theorem 3). Our approximation result is proved in Section 5 (Theorem 4). In the sixth and last section we present two consequences of Theorem 4.

2. Preliminaries

2.1. Some facts about Legendre functions. Legendre functions mapping a general Banach space X into $(-\infty, +\infty]$ are defined in [**3**]. According to [**3**, Theorems 5.4 and 5.6], since X reflexive, the function f is Legendre if and only if it satisfies the following two conditions:

(L1) The interior of the domain of f, int dom f, is nonempty, f is Gâteaux differentiable (see below) on int dom f, and

$$\operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f;$$

(L2) The interior of the domain of f^*, int dom f^*, is nonempty, f^* is Gâteaux differentiable on int dom f^*, and

$$\operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*.$$

Since X is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [**7**, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},$$

$$\operatorname{ran} \nabla f = \operatorname{dom}\ \nabla f^* = \operatorname{int} \operatorname{dom} f^*$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom}\ \nabla f = \operatorname{int} \operatorname{dom} f.$$

Also, conditions (L1) and (L2), in conjunction with [**3**, Theorem 5.4], imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Several interesting examples of Legendre functions are presented in [**2**] and [**3**]. Among them are the functions $\frac{1}{s} \|\cdot\|^s$ with $s \in (1, \infty)$, where the Banach space X is smooth and strictly convex and, in particular, a Hilbert space.

2.2. Two properties of gradients. For any convex $f : X \to (-\infty, +\infty]$ we denote by dom f the set $\{x \in X : f(x) < +\infty\}$. For any $x \in \operatorname{int} \operatorname{dom} f$ and $y \in X$, we denote by $f^\circ(x, y)$ the *right-hand derivative of f at x* in the direction y, that is,

$$f^{\circ}(x,y) := \lim_{t \searrow 0} \frac{f(x+ty)-f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable at* x if $\lim_{t\to 0}\left(f(x+ty)-f(x)\right)/t$ exists for any y. The function f is said to be *Fréchet differentiable at* x if this limit is attained uniformly in $\|y\|=1$. Finally, f is said to be *uniformly Fréchet differentiable on a subset* E *of* X if the limit is attained uniformly for $x \in E$ and $\|y\|=1$. We will need the following result.

Proposition 1 (*cf.* [**27**, Proposition 2.1, p. 474]). *If* $f : X \to \mathbb{R}$ *is uniformly Fréchet differentiable and bounded on bounded subsets of* X*, then* ∇f *is uniformly continuous on bounded subsets of* X *from the strong topology of* X *to the strong topology of* X^**.*

Proposition 2. *If* $f : X \to \mathbb{R}$ *is a positively homogeneous function of degree* $\alpha \in \mathbb{R}$*, then* ∇f *is a positively homogeneous function of degree* $\alpha - 1$*.*

Proof. By the definition of the gradient we have

$$\begin{aligned}\nabla f\left(tx\right) &= \lim_{h\to 0}\frac{f\left(tx+hy\right)-f\left(tx\right)}{h} = \lim_{h\to 0}\frac{f\left(tx+thy\right)-f\left(tx\right)}{th} \\ &= \frac{t^{\alpha}}{t}\lim_{h\to 0}\frac{f\left(x+hy\right)-f\left(x\right)}{h} = t^{\alpha-1}\nabla f\left(x\right)\end{aligned}$$

for any $x \in X$ and all $t > 0$. □

2.3. Some facts about totally convex functions. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0,+\infty]$, defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y-x \rangle, \tag{2.1}$$

is called the *Bregman distance with respect to* f (*cf.* [**16**]). With the function f we associate the function $W^f : X^* \times X \to [0,+\infty]$ defined by

$$W^f(\xi,x) = f(x) - \langle \xi, x \rangle + f^*(\xi).$$

It is clear that $W^f(\nabla f(x), y) = D_f(y,x)$ for any $x, y \in \operatorname{int}\operatorname{dom} f$.

The Bregman distance has the following important properties, called the *three point identity*: for any $x, y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle \,, \tag{2.2}$$

and the *four point identity*: for any $x, y, z, w \in \operatorname{int}\operatorname{dom} f$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle \,. \tag{2.3}$$

Recall that, according to [**13**, Section 1.2, p. 17] (see also [**12**]), the function f is called *totally convex at a point* $x \in \operatorname{int}\operatorname{dom} f$ if its *modulus of total convexity at* x, that is, the function $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x, t) := \inf \{ D_f(y, x) \ : \ y \in \operatorname{dom} f, \ \ \|y - x\| = t \} \,,$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int}\operatorname{dom} f$. Examples of totally convex functions can be found, for instance, in [**13, 15**]. The next proposition turns out to be very useful in the proof of Theorem 4 below.

Proposition 3 (*cf.* [**28**, Proposition 2.2, p. 3]). *If* $x \in \operatorname{int}\operatorname{dom} f$*, then the following statements are equivalent:*

(i) The function f *is totally convex at* x*;*

(ii) For any sequence $\{y_n\}_{n \in \mathbb{N}} \subset \operatorname{dom} f$*,*

$$\lim_{n \to +\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n \to +\infty} \|y_n - x\| = 0.$$

2.4. Some facts about Bregman firmly nonexpansive mappings. Let C be a nonempty, closed and convex subset of $\operatorname{int}\operatorname{dom} f$. We say that a mapping $T : C \to C$ is a *Bregman firmly nonexpansive mapping with respect to* f *(BFNE with respect to* f for short*)* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \tag{2.4}$$

for all $x, y \in C$. It is clear from the definition of the Bregman distance (2.1) that (2.4) is equivalent to

(2.5)
$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

Bauschke, Borwein and Combettes [**4**, Prop. 3.8, p. 604] prove that the resolvent $\mathrm{Res}_A^f = (\nabla f + A)^{-1} \circ \nabla f$ is a BFNE mapping with respect to f whenever A is a monotone mapping.

We remark in passing that an analogous result for very general resolvents can be found in a recent paper by Bauschke, Wang and Yao [**5**].

2.5. The resolvent of A relative to f. Let $A : X \to 2^{X^*}$ be an operator and assume that f is Gâteaux differentiable. The operator

$$\mathrm{Prt}_A^f := (\nabla f + A)^{-1} : X^* \to 2^X$$

is called the *protoresolvent* of A, or, more precisely, the *protoresolvent of A relative to f*. This allows us to define the *resolvent* of A, or, more precisely, the *resolvent of A relative to f*, introduced and studied in [**4**], as the operator $\mathrm{Res}_A^f : X \to 2^X$ given by $\mathrm{Res}_A^f := \mathrm{Prt}_A^f \circ \nabla f$. This operator is single-valued when A is monotone and f is strictly convex on $\operatorname{int}\operatorname{dom} f$. If $A = \partial\varphi$, where φ is a proper, lower semicontinuous and convex function, then we denote

$$\mathrm{Prox}_\varphi^f := \mathrm{Prt}_{\partial\varphi}^f \quad \text{and} \quad \mathrm{prox}_\varphi^f := \mathrm{Res}_{\partial\varphi}^f.$$

If C is a nonempty, closed and convex subset of X, then the indicator function ι_C of C, that is, the function

$$\iota_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C, \end{cases}$$

is proper, convex and lower semicontinuous, and therefore $\partial\iota_C$ exists and is a maximal monotone operator with domain C. The operator $\mathrm{prox}_{\iota_C}^f$ is called the *Bregman* *projection* onto C with respect to f (*cf.* [**8**]) and we denote it by proj_C^f. Note that if X is a Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$, then the Bregman projection of x onto C, *i.e.*, $\operatorname{argmin}\{\|y-x\| : y \in C\}$, is the metric projection P_C.

Recall that the Bregman projection of x onto the nonempty, closed and convex set $K \subset \operatorname{dom} f$ is the necessarily unique vector $\mathrm{proj}_K^f(x) \in K$ satisfying

$$D_f\left(\mathrm{proj}_K^f(x), x\right) = \inf\{D_f(y,x) : y \in K\}.$$

Similarly to the metric projection in Hilbert spaces, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Proposition 4 (*cf.* [**15**, Corollary 4.4, p. 23]). *Suppose that f is totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following conditions are equivalent*:

(i) The vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0, \qquad \forall y \in K;$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \qquad \forall y \in K.$$

3. Two properties of Bregman firmly nonexpansive mappings

In this section we present two properties of the fixed point set $F(T)$ of a BFNE mapping. We first show that $F(T)$ is closed and convex for any BFNE mapping with respect to f when f is also Gâteaux differentiable.

Lemma 1. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$, *and let $T : C \to C$ be a BFNE mapping with respect to f. Then $F(T)$ is closed and convex.*

Proof. It is sufficient to consider the case where $F(T)$ is nonempty. From (2.5) it follows that

$$D_f(x, Ty) + D_f(Ty, y) \leq D_f(x, y)$$

for any $x \in F(T)$ and $y \in C$. *A fortiori*,

$$D_f(x, Ty) \leq D_f(x, y) \tag{3.1}$$

for any $x \in F(T)$ and $y \in C$.

We first show that $F(T)$ is closed. To this end, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $F(T)$ such that $x_n \to \bar{x}$. From (3.1) it follows that

$$D_f(x_n, T\bar{x}) \leq D_f(x_n, \bar{x}) \tag{3.2}$$

for any $n \in \mathbb{N}$. Since f is continuous at $\bar{x} \in C \subset \operatorname{int}\operatorname{dom} f$ and $x_n \to \bar{x}$, it follows that

$$\begin{aligned}\lim_{n\to+\infty} D_f(x_n, T\bar{x}) &= \lim_{n\to+\infty} \left[f(x_n) - f(T\bar{x}) - \langle \nabla f(T\bar{x}), x_n - T\bar{x}\rangle\right] \\ &= \left[f(\bar{x}) - f(T\bar{x}) - \langle \nabla f(T\bar{x}), \bar{x} - T\bar{x}\rangle\right] = D_f(\bar{x}, T\bar{x})\end{aligned}$$

and

$$\lim_{n\to+\infty} D_f(x_n, \bar{x}) = D_f(\bar{x}, \bar{x}) = 0.$$

Thus (3.2) implies that $D_f(\bar{x}, T\bar{x}) = 0$ and therefore it follows from [**3**, Lemma 7.3(vi), p. 642] that $\bar{x} = T\bar{x}$. Hence $\bar{x} \in F(T)$ and this means that $F(T)$ is closed, as claimed.

Next we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0, 1)$, put $z = tx + (1-t)y$. We have to show that $Tz = z$. Indeed, from the definition of the

Bregman distance and (3.1) it follows that

$$\begin{aligned} D_f(z,Tz) &= f(z) - f(Tz) - \langle \nabla f(Tz), z - Tz \rangle \\ &= f(z) - f(Tz) - \langle \nabla f(Tz), tx + (1-t)y - Tz \rangle \\ &= f(z) + tD_f(x,Tz) + (1-t)D_f(y,Tz) - tf(x) - (1-t)f(y) \\ &\le f(z) + tD_f(x,z) + (1-t)D_f(y,z) - tf(x) - (1-t)f(y) \\ &= \langle \nabla f(z), z - tx - (1-t)y \rangle = 0. \end{aligned}$$

Again from [**3**, Lemma 7.3(vi), p. 642] it follows that $Tz = z$. Therefore $F(T)$ is also convex, as asserted. □

Next we show that if f is a Legendre function which is uniformly Fréchet differentiable on bounded subsets of X, and T is a BFNE mapping with respect to f, then the set of fixed points of T coincides with the set of its asymptotic fixed points. Recall that a point $u \in C$ is said to be an *asymptotic fixed point* [**26**] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$.

Lemma 2. *Let $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of X. Let C be a nonempty, closed and convex subset of X and let $T : C \to C$ be a BFNE mapping with respect to f. Then $F(T) = \hat{F}(T)$.*

Proof. The inclusion $F(T) \subset \hat{F}(T)$ is obvious. To show that $F(T) \supset \hat{F}(T)$, let $u \in \hat{F}(T)$ be given. Then we have a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since f is uniformly Fréchet differentiable on bounded subsets of X, ∇f is uniformly continuous on bounded subsets of X (see Proposition 1). Hence $(\nabla f(Tx_n) - \nabla f(x_n)) \to 0$ as $n \to +\infty$ and therefore

$$\lim_{n\to+\infty} \langle \nabla f(Tx_n) - \nabla f(x_n), y \rangle = 0 \tag{3.3}$$

for any $y \in X$, and

$$\lim_{n\to+\infty} \langle \nabla f(Tx_n) - \nabla f(x_n), x_n \rangle = 0, \tag{3.4}$$

because $\{x_n\}_{n\in\mathbb{N}}$ is bounded. On the other hand, since T is a BFNE mapping with respect to f, we have

$$0 \le D_f(Tx_n, u) - D_f(Tx_n, Tu) + D_f(Tu, x_n) - D_f(Tu, Tx_n). \tag{3.5}$$

From the three point identity (2.2) and (3.5) we now obtain

$$\begin{aligned} D_f(u, Tu) &= D_f(Tx_n, Tu) - D_f(Tx_n, u) - \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle \\ &\le D_f(Tu, x_n) - D_f(Tu, Tx_n) - \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle \\ &= [f(Tu) - f(x_n) - \langle \nabla f(x_n), Tu - x_n \rangle] - \\ &[f(Tu) - f(Tx_n) - \langle \nabla f(Tx_n), Tu - Tx_n \rangle] \\ &- \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle \end{aligned}$$

$$= f(Tx_n) - f(x_n) - \langle \nabla f(x_n), Tu - x_n \rangle + \langle \nabla f(Tx_n), Tu - Tx_n \rangle$$

$$- \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle$$

$$= - [f(x_n) - f(Tx_n) - \langle \nabla f(Tx_n), x_n - Tx_n \rangle] - \langle \nabla f(Tx_n), x_n - Tx_n \rangle$$

$$- \langle \nabla f(x_n), Tu - x_n \rangle + \langle \nabla f(Tx_n), Tu - Tx_n \rangle$$

$$- \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle$$

$$= -D_f (x_n, Tx_n) - \langle \nabla f(Tx_n), x_n - Tx_n \rangle - \langle \nabla f(x_n), Tu - x_n \rangle$$

$$+ \langle \nabla f(Tx_n), Tu - Tx_n \rangle - \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle$$

$$\leq - \langle \nabla f(Tx_n), x_n - Tx_n \rangle - \langle \nabla f(x_n), Tu - x_n \rangle$$

$$+ \langle \nabla f(Tx_n), Tu - Tx_n \rangle - \langle \nabla f(u) - \nabla f(Tu), Tx_n - u \rangle$$

$$= \langle \nabla f(x_n) - \nabla f(Tx_n), x_n - Tu \rangle - \langle \nabla f(u) - \nabla f(Tu), Tx_n - x_n \rangle$$

$$- \langle \nabla f(u) - \nabla f(Tu), x_n - u \rangle .$$

From (3.3), (3.4), and the hypotheses $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ we get that $D_f (u, Tu) \leq 0$. Consequently, $D_f (u, Tu) = 0$ and from [**3**, Lemma 7.3(vi), p. 642] it follows that $Tu = u$. That is, $u \in \mathrm{F}(T)$, as required. □

4. Existence of Fixed Points

In this section we obtain necessary and sufficient conditions for BFNE mappings to have a (common) fixed point in general reflexive Banach spaces. We begin with a theorem for a single BFNE mapping. This result can be proved by combining Theorem 3.3 and Lemma 7.3(viii) of [**3**] with Proposition 4.1(v)(a) of [**4**]. However, we include a more detailed version of the proof for the readers convenience.

Theorem 1. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\operatorname{int} \operatorname{dom} f^*$*. Let C be a nonempty, closed and convex subset of* $\operatorname{int} \operatorname{dom} f$ *and let $T : C \to C$ be a BFNE mapping with respect to f. If $F(T)$ is nonempty, then $\{T^n y\}_{n \in \mathbb{N}}$ is bounded for each $y \in C$.*

Proof. We know by (3.1) that

$$D_f(x, Ty) \leq D_f(x, y)$$

for any $x \in F(T)$ and $y \in C$. Therefore

$$D_f(x, T^n y) \leq D_f(x, y)$$

for any $x \in F(T)$ and $y \in C$. This inequality shows that the nonnegative sequence $\{D_f(x, T^n y)\}_{n\in\mathbb{N}}$ is bounded. Let M be an upper bound of $\{D_f(x, T^n y)\}_{n\in\mathbb{N}}$. Then

$$f(x) - \langle \nabla f(T^n y), x\rangle + f^*(\nabla f(T^n y)) = W^f(\nabla f(T^n y), x) = D_f(x, T^n y) \leq M.$$

This implies that the sequence $\{\nabla f(T^n y)\}_{n\in\mathbb{N}}$ is contained in the sublevel set $\operatorname{lev}^{\psi}_{\leq}(M - f(x))$ of the function $\psi = f^* - \langle \cdot, x\rangle$. Since the function f^* is proper and lower semicontinuous, an application of the Moreau-Rockafellar Theorem [**29**, Theorem 7A] shows that $\psi = f^* - \langle \cdot, x\rangle$ is coercive. Consequently, all sublevel sets of ψ are bounded. Hence, the sequence $\{\nabla f(T^n y)\}_{n\in\mathbb{N}}$ is bounded. Since the function f^* is bounded on bounded subsets of X by hypothesis, the gradient ∇f^* is also bounded on bounded subsets of X [**13**, Proposition 1.1.11, p. 17]. Thus the sequence $T^n y = \nabla f^*(\nabla f(T^n y))$, $n \in \mathbb{N}$, is bounded too, as claimed. □

For a mapping $T : C \to C$, let $S_n(z) := 1/n \sum_{k=1}^{n} T^k z$ for all $z \in C$.

Theorem 2. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and let $T : C \to C$ be a BFNE mapping with respect to f. If there exists $y \in C$ such that $\|S_n(y)\| \nrightarrow \infty$ as $n \to \infty$, then $F(T)$ is nonempty.*

Proof. Suppose that there exists $y \in C$ such that $\|S_n(y)\| \nrightarrow \infty$ as $n \to \infty$. Let $x \in C$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be given. Since T is BFNE with respect to f, we have

$$D_f\left(T^{k+1}y, Tx\right) + D_f\left(Tx, T^{k+1}y\right) \leq D_f\left(Tx, T^k y\right) + D_f\left(T^{k+1}y, x\right). \tag{4.1}$$

From the three point identity (2.2) we get that

$$D_f\left(T^{k+1}y, Tx\right) + D_f\left(Tx, T^{k+1}y\right) \leq D_f\left(Tx, T^k y\right) + D_f\left(T^{k+1}y, Tx\right) + D_f\left(Tx, x\right) + \left\langle \nabla f(Tx) - \nabla f(x), T^{k+1}y - Tx \right\rangle.$$

This implies that

$$0 \leq D_f\left(Tx, x\right) + D_f\left(Tx, T^k y\right) - D_f\left(Tx, T^{k+1}y\right) + \left\langle \nabla f(Tx) - \nabla f(x), T^{k+1}y - Tx \right\rangle.$$

Summing these inequalities with respect to $k = 0, 1, \ldots, n-1$, we now obtain

$$0 \leq nD_f\left(Tx, x\right) + D_f\left(Tx, y\right) - D_f\left(Tx, T^n y\right) + \left\langle \nabla f(Tx) - \nabla f(x), \sum_{k=0}^{n-1} T^{k+1}y - nTx \right\rangle.$$

Dividing this inequality by n, we have

$$0 \leq D_f\left(Tx, x\right) + \frac{1}{n}\left[D_f\left(Tx, y\right) - D_f\left(Tx, T^n y\right)\right] + \left\langle \nabla f(Tx) - \nabla f(x), S_n(y) - Tx \right\rangle \tag{4.2}$$

and

$$0 \leq D_f\left(Tx, x\right) + \frac{1}{n}D_f\left(Tx, y\right) + \left\langle \nabla f(Tx) - \nabla f(x), S_n(y) - Tx \right\rangle. \tag{4.3}$$

Since $\|S_n(y)\| \nrightarrow \infty$ as $n \to \infty$ by assumption, there exists a subsequence $\left\{S_{n_k}(y)\right\}_{k\in\mathbb{N}}$ of $\left\{S_n(y)\right\}_{n\in\mathbb{N}}$ such that $S_{n_k}(y) \rightharpoonup u \in C$. Letting $n_k \to +\infty$ in (4.3), we obtain

$$0 \leq D_f\left(Tx, x\right) + \left\langle \nabla f(Tx) - \nabla f(x), u - Tx \right\rangle. \tag{4.4}$$

Setting $x = u$ in (4.4), we get from the four point identity (2.3) that

$$\begin{aligned} 0 &\leq D_f(Tu, u) + \langle \nabla f(Tu) - \nabla f(u), u - Tu \rangle \\ &= D_f(Tu, u) + D_f(u, u) - D_f(u, Tu) - D_f(Tu, u) + D_f(Tu, Tu) \\ &= -D_f(u, Tu). \end{aligned}$$

Hence $D_f(u, Tu) \leq 0$ and so $D_f(u, Tu) = 0$. It now follows from [**3**, Lemma 7.3(vi), p. 642] that $Tu = u$. That is, $u \in F(T)$. This completes the proof of Theorem 2. □

Remark 1. As can be seen from the proof, Theorem 2 remains true for those mappings which only satisfy (4.1). In the special case where $f = 1/2 \left\|\cdot\right\|^2$, such mappings are called *non-spreading*. For more information see [**21**].

Remark 2. We remark in passing that we still do not know if the analog of Theorem 2 for nonexpansive mappings holds outside Hilbert space (*cf.* [**24**, Remark 2, p. 275]).

Corollary 1. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Every nonempty, bounded, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *has the fixed point property for BFNE self-mappings with respect to f.*

As in [**21**], Corollary 1, when combined with Lemma 1, yields the following result.

Theorem 3. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty, bounded, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$. *Let $\{T_\alpha\}_{\alpha \in A}$ be a commutative family of BFNE mappings with respect to f from C into itself. Then the family $\{T_\alpha\}_{\alpha \in A}$ has a common fixed point.*

5. Approximation of Fixed Points

In this section we prove a strong convergence theorem of Browder's type for BFNE mappings with respect to a well chosen function f.

Theorem 4. *Let $f : X \to \mathbb{R}$ be a Legendre, totally convex function which is positively homogeneous of degree $\alpha > 1$, uniformly Fréchet differentiable and* *bounded on bounded subsets of X. Let C be a nonempty, bounded, closed and convex subset of X with $0 \in C$, and let T be a BFNE self-mapping with respect to f. Then the following two assertions hold:*

(i) For each $t \in (0,1)$, there exists a unique $u_t \in C$ satisfying $u_t = tTu_t$;

(ii) The net $\{u_t\}_{t \in (0,1)}$ converges strongly to $\mathrm{proj}^f_{F(T)}(\nabla f^*(0))$ as $t \to 1^-$.

Proof. (i) Fix $t \in (0,1)$ and let S_t be the mapping defined by $S_t = tT$. Since $0 \in C$ and C is convex, S_t is a mapping from C into itself. We next show that S_t is a BFNE mapping with respect to f. Indeed, if $x, y \in C$, then, since T is BFNE with respect to f, it follows from Proposition 2 that

$$\begin{aligned} \langle \nabla f(S_t x) - \nabla f(S_t y), S_t x - S_t y\rangle &= t^{\alpha} \langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\leq t^{\alpha} \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \\ &= t^{\alpha-1} \langle \nabla f(x) - \nabla f(y), S_t x - S_t y\rangle \\ &\leq \langle \nabla f(x) - \nabla f(y), S_t x - S_t y\rangle . \end{aligned} \tag{5.1}$$

Thus S_t is also BFNE with respect to f. Since C is bounded, it follows from Corollary 1 that S_t has a fixed point. We next show that $F(S_t)$ consists of exactly one point. If $u, u' \in F(S_t)$, then it follows from (5.1) that

$$\begin{aligned} \langle \nabla f(u) - \nabla f(u'), u - u'\rangle &= \langle \nabla f(S_t u) - \nabla f(S_t u'), S_t u - S_t u'\rangle \\ &\leq t^{\alpha-1} \langle \nabla f(u) - \nabla f(u'), S_t u - S_t u'\rangle \\ &= t^{\alpha-1} \langle \nabla f(u) - \nabla f(u'), u - u'\rangle . \end{aligned} \tag{5.2}$$

By (5.2) and the monotonicity of ∇f, we have

$$\langle \nabla f(u) - \nabla f(u'), u - u'\rangle = 0.$$

Since f is Legendre, ∇f is strictly monotone and therefore $u = u'$. Thus there exists a unique $u_t \in C$ such that $u_t = S_t u_t$.

(ii) Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $t_n \to 1^-$ as $n \to +\infty$. Put $x_n = u_{t_n}$ for all $n \in \mathbb{N}$. By Lemma 1 and Theorem 2, $F(T)$ is nonempty, closed and convex. Thus the Bregman projection $\mathrm{proj}^f_{F(T)}$ is well defined. In order to show that $u_t \to \mathrm{proj}^f_{F(T)}(\nabla f^*(0))$, it is sufficient to show that $x_n \to \mathrm{proj}^f_{F(T)}(\nabla f^*(0))$. Since C is bounded, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup v$. By the definition of x_n, we have $\|x_n - Tx_n\| = (1-t_n)\|Tx_n\|$ for all $n \in \mathbb{N}$. So, we have $x_n - Tx_n \to 0$ and hence $v \in \hat{F}(T)$. Lemma 2 now implies that $v \in F(T)$. We next show that $x_{n_k} \to v$. Let $y \in F(T)$ be given and fix $n \in \mathbb{N}$. Then, since T is BFNE with respect to f, we have

$$\langle \nabla f(Tx_n) - \nabla f(Ty), Tx_n - Ty\rangle \le \langle \nabla f(x_n) - \nabla f(y), Tx_n - Ty\rangle .$$

That is,

$$0 \le \langle \nabla f(x_n) - \nabla f(Tx_n), Tx_n - y\rangle .$$

Since

$$\begin{aligned}\nabla f(x_n) - \nabla f(Tx_n) &= \nabla f(t_n Tx_n) - \nabla f(Tx_n)\\ &= t_n^{\alpha-1}\nabla f(Tx_n) - \nabla f(Tx_n) = (t_n^{\alpha-1}-1)\nabla f(Tx_n),\end{aligned}$$

we have

$$0 \le \left\langle (t_n^{\alpha-1}-1)\nabla f(Tx_n), Tx_n - y\right\rangle .$$

This yields

$$0 \le \langle -\nabla f(Tx_n), Tx_n - y\rangle \tag{5.3}$$

and

$$\langle \nabla f(y) - \nabla f(Tx_n), y - Tx_n\rangle \le \langle \nabla f(y), y - Tx_n\rangle . \tag{5.4}$$

Since $x_{n_k} \rightharpoonup v$ and $x_{n_k} - Tx_{n_k} \to 0$, it follows that $Tx_{n_k} \rightharpoonup v$. Hence from (5.4) we obtain that

$$\limsup_{k\to+\infty} \langle \nabla f(y) - \nabla f(Tx_{n_k}), y - Tx_{n_k} \rangle \leq \limsup_{k\to+\infty} \langle \nabla f(y), y - Tx_{n_k} \rangle = \langle \nabla f(y), y - v \rangle . \tag{5.5}$$

Substituting $y = v$ in (5.5), we get

$$0 \leq \limsup_{k\to+\infty} \langle \nabla f(v) - \nabla f(Tx_{n_k}), v - Tx_{n_k} \rangle \leq 0.$$

Thus

$$\lim_{k\to+\infty} \langle \nabla f(v) - \nabla f(Tx_{n_k}), v - Tx_{n_k} \rangle = 0.$$

Since

$$D_f(v, Tx_{n_k}) + D_f(Tx_{n_k}, v) = \langle \nabla f(v) - \nabla f(Tx_{n_k}), v - Tx_{n_k} \rangle$$

it follows that

$$\lim_{k\to+\infty} D_f(v, Tx_{n_k}) = \lim_{k\to+\infty} D_f(Tx_{n_k}, v) = 0.$$

Proposition 3 now implies that $Tx_{n_k} \to v$. Finally, we claim that $v = \operatorname{proj}^f_{F(T)}(\nabla f^*(0))$. Since ∇f is norm-to-weak* continuous on bounded subsets, it follows that $\nabla f(Tx_{n_k}) \rightharpoonup \nabla f(v)$. Setting $n := n_k$ and letting $k \to +\infty$ in (5.3), we obtain

$$0 \leq \langle -\nabla f(v), v - y \rangle$$

for any $y \in F(T)$. Hence

$$0 \leq \langle \nabla f(\nabla f^*(0)) - \nabla f(v), v - y \rangle$$

for any $y \in F(T)$. Thus Proposition 4 implies that $v = \operatorname{proj}^f_{F(T)}(\nabla f^*(0))$. Consequently, the whole net $\{u_t\}_{t\in(0,1)}$ converges strongly to $\operatorname{proj}^f_{F(T)}(\nabla f^*(0))$ as $t \to 1^-$. This completes the proof of Theorem 4. □

Remark 3. Early analogs of Theorem 4 for nonexpansive mappings in Hilbert and Banach spaces may be found in [**9, 18, 25**].

6. Consequences of the Approximation Result

We first specialize Theorem 4 to the case where $f(x) = \frac{1}{2}\|x\|^2$ and X is a uniformly smooth and uniformly convex Banach space, and then apply it to the problem of finding zeroes of a maximal monotone operator $A : X \to 2^{X^*}$. In this case the function $f(x) = \frac{1}{2}\|x\|^2$ is Legendre (*cf.* [**3**, Lemma 6.2, p.24]) and uniformly Fréchet differentiable on bounded subsets of X. According to [**14**, Corollary 1(ii), p. 325], since X is uniformly convex, f is totally convex. Thus we obtain the following corollary.

Corollary 2. *Let X be a uniformly smooth and uniformly convex Banach space. Let C be a nonempty, bounded, closed and convex subset of X with $0 \in C$, and let $T : C \to C$ be of firmly nonexpansive type. Then the following two assertions hold:*

(i) For each $t \in (0, 1)$, there exists a unique $u_t \in C$ satisfying $u_t = tTu_t$;

(ii) The net $\{u_t\}_{t\in(0,1)}$ converges strongly to $\operatorname{proj}^f_{F(T)}(0)$ as $t \to 1^-$.

As a matter of fact, this corollary is known to hold even when X is only a smooth and uniformly convex Banach space [**21**].

As a direct consequence of Theorem 4 we get the following new result.

Corollary 3. *Let $f : X \to \mathbb{R}$ be a Legendre, totally convex function which is positively homogeneous of degree $\alpha > 1$, uniformly Fréchet differentiable and bounded on bounded subsets of X. Let C be a nonempty, bounded, closed and convex subset of X with $0 \in C$. Let λ be positive real number and let A be a monotone operator such that $\operatorname{dom} A \subset C \subset (\nabla f)^{-1}(\operatorname{ran}(\nabla f + \lambda A))$. Then the following two assertions hold:*

(i) For each $t \in (0, 1)$, there exists a unique $u_t \in C$ satisfying $u_t = t\operatorname{Res}^f_{\lambda A} u_t$;

(ii) The net $\{u_t\}_{t\in(0,1)}$ converges strongly to $\operatorname{proj}^f_{A^{-1}(0^*)}(\nabla f^*(0))$ as $t \to 1^-$.

Remark 4. Algorithm 5.5 in [**6**] provides another way for constructing Bregman projections onto the zero sets of maximal monotone operators.

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