A Projection Method for Solving Nonlinear Problems in Reflexive Banach Spaces

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Abstract. We study the convergence of an iterative algorithm for finding common fixed points of finitely many Bregman firmly nonexpansive operators in reflexive Banach spaces. Our algorithm is based on the concept of the so-called shrinking projection method and takes into account possible computational errors. We establish a strong convergence theorem and then apply it to the solution of convex feasibility and equilibrium problems, and to finding zeroes of two different classes of nonlinear mappings.

1. Introduction

Let X denote a real reflexive Banach space with norm $\|\cdot\|$ and let X^* stand for the (topological) dual of X equipped with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x \rangle$.

In this paper $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and convex function, and $f^* : X^* \to (-\infty, +\infty]$ is the Fenchel conjugate of f. We denote by $\operatorname{dom} f$ the domain of f, that is, the set $\{x \in X : f(x) < +\infty\}$. The set of nonnegative integers is denoted by $\mathbb{N}$.

Let $f : X \to (-\infty, +\infty]$ be Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f$, the interior of $\operatorname{dom} f$ (see Section 2.1), and let K be a nonempty, closed and convex subset of

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int dom f. An operator $T : K \to K$ is said to be *Bregman firmly nonexpansive* (see [**5**] and Section 2.5) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \tag{1.1}$$

for all $x, y \in K$, where ∇f is the gradient of f (see also Section 2.1). If the Banach space X is a Hilbert space H and the function $f(x) = (1/2)\|x\|^2$, $x \in H$, then $\nabla f = I$, where I is the identity operator on H. In this case inequality (1.1) is reduced to the following inequality:

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \tag{1.2}$$

for all $x, y \in K$. An operator T which satisfies (1.2) is called *firmly nonexpansive* [**18**, p. 42]. It turns out that firmly nonexpansive fixed point theory in Hilbert space can be applied to the solution of diverse problems, such as finding zeroes of monotone mappings and solving certain evolution equations, as well as convex feasibility (CFP) and equilibrium problems (EP).

When we try to extend this theory to Banach spaces we need to use the Bregman distance (see Section 2.3) and Bregman firmly nonexpansive operators (see Section 2.5 for more details). This is because many of the useful examples of firmly nonexpansive operators in Hilbert space are no longer firmly nonexpansive in Banach spaces (for example, the metric projection P_K onto a nonempty, closed and convex subset K of H). The Bregman projection (Section 2.4) and the generalized resolvent (Section 5) are examples of Bregman firmly nonexpansive operators.

We remark in passing that other types of firmly nonexpansive operators in Banach spaces and in the Hilbert ball are studied, for example, in [**10, 11, 18, 19, 20, 21, 22**] and the references therein.

In some cases it is enough to assume that the operator T is *quasi-Bregman firmly nonexpansive* (see Section 2.5), that is,

$$\langle \nabla f(x) - \nabla f(Tx), Tx - p \rangle \geq 0$$

for all $x \in K$ and $p \in F(T)$, where $F(T)$ stands for the (nonempty) fixed point set of T.

In this paper we are concerned with quasi-Bregman firmly nonexpansive operators (see Section 2.5). Our main goal is to study the convergence of an iterative algorithm for finding common fixed points of finitely many quasi-Bregman firmly nonexpansive operators in reflexive Banach spaces. This algorithm is based on the concept of the so-called shrinking projection method, which was introduced by Takahashi, Takeuchi and Kubota in [**30**]. Our algorithm allows for possible computational errors. We establish a strong convergence theorem (Theorem 1 below) and then get as corollaries methods for solving convex feasibility problems (Corollary 1), finding zeroes of maximal monotone mappings (Corollary 2) and solving equilibrium problems (Corollary 3). All these corollaries also take into account possible computational errors. In addition, we obtain a method for finding zeroes of Bregman inverse strongly monotone mappings (Corollary 4). Other algorithms which can be applied to BFNE operators can be found, for example, in [**6, 24, 25**].

The paper is organized as follows. In Section 2 we present several preliminary definitions and results. The third section is devoted to the study of our iterative method. In Sections 4–7 we modify this method in order to solve other problems: convex feasibility problems (Section 4), finding zeroes of maximal monotone mappings (Section 5), equilibrium problems (Section 6) and finding zeroes of Bregman inverse strongly monotone mappings (Section 7). For more information regarding these problems see, for example, [**2**], [**29**], [**7**] and [**14**], respectively. In Section 8 we observe that our methods may also be used for finding common solutions to mixed problems.

2. Preliminaries

2.1. A property of gradients. Let $f : X \to (-\infty, +\infty]$ be convex. For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in X$, we denote by $f^{\circ}(x, y)$ the *right-hand derivative of* f *at*

x in the direction y, that is,

$$f^{\circ}(x,y) := \lim_{t \searrow 0} \frac{f(x+ty) - f(x)}{t}.$$

The function f is called *Gâteaux differentiable at* x if $\lim_{t \to 0} \left(f(x+ty) - f(x) \right)/t$ exists for any y. In this case $f^{\circ}(x,y)$ coincides with $(\nabla f)(x)$, the value of the *gradient* ∇f of f at x. The function f is said to be *Fréchet differentiable at* x if this limit is attained uniformly for $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable on a subset* E *of* X if the limit is attained uniformly for $x \in E$ and $\|y\| = 1$. We will need the following result.

Proposition 1 (*cf.* [**24**, Proposition 2.1, p. 474]). *If* $f : X \to \mathbb{R}$ *is uniformly Fréchet differentiable and bounded on bounded subsets of* X*, then* ∇f *is uniformly continuous on bounded subsets of* X *from the strong topology of* X *to the strong topology of* X^*.

2.2. Some facts about Legendre functions. Legendre functions which map a general Banach space X into $(-\infty, +\infty]$ are defined in [**4**]. According to [**4**, Theorems 5.4 and 5.6], since X is reflexive, the function f is *Legendre* if and only if it satisfies the following two conditions:

(L1) The interior of the domain of f, $\operatorname{int} \operatorname{dom} f$, is nonempty, f is Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f$, and

$$\operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f;$$

(L2) The interior of the domain of f^*, $\operatorname{int} \operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int} \operatorname{dom} f^*$, and

$$\operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*.$$

Since X is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [**8**, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},$$

$$\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f.$$

Also, conditions (L1) and (L2), in conjunction with [**4**, Theorem 5.4], imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Several interesting examples of Legendre functions are presented in [**3**] and [**4**]. Among them are the functions $\frac{1}{s}\|\cdot\|^s$ with $s \in (1, \infty)$, where the Banach space X is smooth and strictly convex and, in particular, a Hilbert space. From now on we assume that the convex function $f : X \to (-\infty, +\infty]$ is Legendre.

2.3. Some facts about the Bregman distance. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$, defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \tag{2.1}$$

is called the *Bregman distance with respect to* f (*cf.* [**16**]). The Bregman distance has the following two important properties, called the *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int} \operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle, \tag{2.2}$$

and the *four point identity*: for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int} \operatorname{dom} f$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle. \tag{2.3}$$

2.4. Some facts about totally convex functions. Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that, according to [**13**, Section 1.2, p. 17] (see also [**12**]), the function f is called *totally convex at a point* $x \in \operatorname{int} \operatorname{dom} f$ if its *modulus of total convexity at* x, that is, the function $\upsilon_f : \operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x, t) := \inf \{D_f(y, x) : y \in \operatorname{dom} f, \quad \|y - x\| = t\},$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int}\operatorname{dom} f$. In addition, the function f is called *totally convex on bounded sets* if $\upsilon_f(E,t)$ is positive for any nonempty bounded subset E of X and for any $t > 0$, where the *modulus of total convexity of the function* f on the set E is the function $\upsilon_f(E,\cdot) : [0,+\infty) \to [0,+\infty]$ defined by

$$\upsilon_f(E,t) := \inf\left\{\upsilon_f(x,t) : x \in E \cap \operatorname{int}\operatorname{dom} f\right\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [**15**, Theorem 2.10, p. 9]).

Examples of totally convex functions can be found, for instance, in [**13, 15**].

The next proposition turns out to be very useful in the proof of Theorem 1 below.

Proposition 2 (*cf.* [**28**, Proposition 2.2, p. 3]). *If* $x \in \operatorname{int}\operatorname{dom} f$, *then the following statements are equivalent:*

(i) The function f *is totally convex at* x*;*

(ii) For any sequence $\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} f$,

$$\lim_{n\to+\infty} D_f\left(y_n, x\right) = 0 \implies \lim_{n\to+\infty} \left\|y_n - x\right\| = 0.$$

Recall that the function f is called *sequentially consistent* (see [**15**]) if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to+\infty} D_f\left(y_n, x_n\right) = 0 \implies \lim_{n\to+\infty} \left\|y_n - x_n\right\| = 0.$$

Proposition 3 (*cf.* [**13**, Lemma 2.1.2, p. 67]). *The function* f *is totally convex on bounded sets if and only if it is sequentially consistent.*

Recall that the *Bregman projection* (*cf.* [**9**]) of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty, closed and convex set $K \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_K^f(x) \in K$ satisfying

$$D_f\left(\operatorname{proj}_K^f(x), x\right) = \inf\left\{D_f\left(y,x\right) : y \in K\right\}.$$

Similarly to the metric projection in Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Proposition 4 (*cf.* [**15**, Corollary 4.4, p. 23]). *Suppose that f is Legendre and totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following conditions are equivalent*:

(i) The vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \qquad \forall y \in K;$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x) \qquad \forall y \in K.$$

The following two propositions exhibit two additional properties of totally convex functions.

Proposition 5 (*cf.* [**25**, Lemma 3.1, p. 31]). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function. If $x_0 \in X$ and the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is bounded, then the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded too.*

Proposition 6 (*cf.* [**25**, Lemma 3.2, p. 31]). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function, $x_0 \in X$, and let K be a nonempty, closed and convex subset of X. Suppose that the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded and any weak subsequential limit of $\{x_n\}_{n \in \mathbb{N}}$ belongs to K. If $D_f(x_n, x_0) \leq D_f\left(\operatorname{proj}_K^f(x_0), x_0\right)$ for any $n \in \mathbb{N}$, then $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $\operatorname{proj}_K^f(x_0)$.*

2.5. Some facts about Bregman firmly nonexpansive operators. A well-known family of operators is the class of Bregman firmly nonexpansive operators, where an operator $T : K \to K$ is called *Bregman firmly nonexpansive (BFNE* for short*)* if

$$\langle \nabla f\left(Tx\right) -\nabla f\left(Ty\right) ,Tx-Ty\rangle \leq \langle \nabla f\left(x\right) -\nabla f\left(y\right) ,Tx-Ty\rangle \tag{2.4}$$

for all $x,y\in K$. It is clear from the definition of the Bregman distance (2.1) that inequality (2.4) is equivalent to

$$D_{f}\left(Tx,Ty\right) +D_{f}\left(Ty,Tx\right) +D_{f}\left(Tx,x\right) +D_{f}\left(Ty,y\right) \leq D_{f}\left(Tx,y\right) +D_{f}\left(Ty,x\right) . \tag{2.5}$$

See [**5, 26**] for more information on BFNE operators. Necessary and sufficient conditions for BFNE operators to have a (common) fixed point in general reflexive Banach spaces can be found in [**26**, Theorems 1.4.1 and 1.4.2]. Recall that a point $u\in K$ is said to be an *asymptotic fixed point* [**23**] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that $x_n\rightharpoonup u$ and $x_n-Tx_n\to 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$. In particular, we prove in [**26**, Lemma 1.3.2] that for any BFNE operator T, $F\left(T\right)=\hat{F}\left(T\right)$ when the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X.

If we take in (2.4) $x\in K$ and $y:=p\in F\left(T\right)$ (assuming that $F\left(T\right)$ is nonempty), then we get the following inequality:

$$\langle \nabla f\left(x\right) -\nabla f\left(Tx\right) ,Tx-p\rangle \geq 0. \tag{2.6}$$

An operator which satisfies inequality (2.6) is called *quasi-Bregman firmly nonexpansive (QBFNE* for short*)*. Again it is clear from the definition of the Bregman distance (2.1) that inequality (2.6) is equivalent to the following one:

$$D_{f}\left(p,Tx\right) +D_{f}\left(Tx,x\right) \leq D_{f}\left(p,x\right) \tag{2.7}$$

for all $x\in K$ and $p\in F\left(T\right)$.

3. A Strong Convergence Theorem

The following algorithm is based on the concept of the so-called shrinking projection method, which was introduced by Takahashi, Takeuchi and Kubota in [**30**]. More precisely, we study the following algorithm when $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$:

$$
\begin{cases}
x_0 \in X, \\
Q_0^i = X, \qquad i = 1, 2, \ldots, N, \\
y_n^i = T_i(x_n + e_n^i), \\
Q_{n+1}^i = \left\{ z \in Q_n^i : \left\langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \right\rangle \leq 0 \right\}, \\
Q_{n+1} := \bigcap_{i=1}^{N} Q_{n+1}^i, \\
x_{n+1} = \operatorname{proj}_{Q_{n+1}}^f(x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases}
\tag{3.1}
$$

Theorem 1. *Let $T_i : X \to X$, $i = 1, 2, \ldots, N$, be N QBFNE operators which satisfy $F(T_i) = \hat{F}(T_i)$ for each $1 \leq i \leq N$ and $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_F^f(x_0)$ as $n \to +\infty$.*

Proof. We divide our proof into four steps.

Step 1. *There are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1).*

Let $n \in \mathbb{N}$. It is not difficult to check that the sets Q_n^i are closed and convex for all $i = 1, 2, \ldots, N$. Hence their intersection Q_n is also closed and convex. Let $u \in F$. For any $n \in \mathbb{N}$, we obtain from (2.6) that

$$\left\langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), u - y_n^i \right\rangle \leq 0,$$

which implies that $u \in Q_{n+1}^i$. Since this holds for any $i = 1, 2, \ldots, N$, it follows that $u \in Q_{n+1}$. Thus $F \subset Q_n$ for any $n \in \mathbb{N}$.

From now on we fix an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (3.1).

Step 2. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded.*

It follows from Proposition 4(iii) that, for each $u \in F$, we have

$$D_f(x_n, x_0) = D_f\left(\mathrm{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f(u, x_0) - D_f\left(u, \mathrm{proj}^f_{Q_n}(x_0)\right) \leq D_f(u, x_0). \tag{3.2}$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in F$. Therefore by Proposition 5 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Step 3. *Every weak subsequential limit of* $\{x_n\}_{n\in\mathbb{N}}$ *belongs to* F.

Since $x_{n+1} \in Q_{n+1} \subset Q_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1}, \mathrm{proj}^f_{Q_n}(x_0)\right) + D_f\left(\mathrm{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f(x_{n+1}, x_0)$$

and hence

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0). \tag{3.3}$$

Therefore the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Step 2), $\lim_{n\to+\infty} D_f(x_n, x_0)$ exists. Thus it follows from (3.3) that

$$\lim_{n\to+\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.4}$$

Proposition 3 now implies that

$$\lim_{n\to+\infty} (x_{n+1} - x_n) = 0. \tag{3.5}$$

For any $i = 1, 2, \ldots, N$, it follows from the definition of the Bregman distance (see (2.1)) that

$$D_f\left(x_n, x_n + e^i_n\right) = f(x_n) - f\left(x_n + e^i_n\right) - \left\langle \nabla f(x_n + e^i_n), x_n - \left(x_n + e^i_n\right)\right\rangle = f(x_n) - f\left(x_n + e^i_n\right) + \left\langle \nabla f(x_n + e^i_n), e^i_n\right\rangle.$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [**13**, Proposition 1.1.11, p. 17]). In addition,

f is uniformly Fréchet differentiable and therefore it is uniformly continuous on bounded subsets (see [**1**, Theorem 1.8, p. 13]). Hence, since $\lim_{n\to+\infty} e_n^i = 0$, we see that

$$\lim_{n\to+\infty} D_f\left(x_n, x_n + e_n^i\right) = 0. \tag{3.6}$$

For any $i = 1, 2, \ldots, N$, it follows from the three point identity (see (2.2)) that

$$\begin{aligned} D_f\left(x_{n+1}, x_n + e_n^i\right) &= D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_n + e_n^i\right) \\ &\quad + \left\langle \nabla f(x_n) - \nabla f(x_n + e_n^i), x_{n+1} - x_n \right\rangle. \end{aligned}$$

Since ∇f is bounded on bounded subsets of X, (3.4), (3.5) and (3.6) imply that

$$\lim_{n\to+\infty} D_f\left(x_{n+1}, x_n + e_n^i\right) = 0. \tag{3.7}$$

For any $i = 1, 2, \ldots, N$, it follows from the inclusion $x_{n+1} \in Q_{n+1}^i$ that

$$\begin{aligned} 0 &\le D_f\left(x_{n+1}, y_n^i\right) + D_f\left(y_n^i, x_n + e_n^i\right) \\ &\le D_f\left(x_{n+1}, y_n^i\right) + D_f\left(y_n^i, x_n + e_n^i\right) + \left\langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), y_n^i - x_{n+1} \right\rangle \\ &= D_f\left(x_{n+1}, x_n + e_n^i\right). \end{aligned}$$

From (3.7) we obtain that

$$\lim_{n\to+\infty} \left(D_f\left(x_{n+1}, y_n^i\right) + D_f\left(y_n^i, x_n + e_n^i\right)\right) = 0.$$

Hence $\lim_{n\to+\infty} D_f\left(x_{n+1}, y_n^i\right) = 0$. Proposition 3 now implies that $\lim_{n\to+\infty}\left(y_n^i - x_{n+1}\right) = 0$. Therefore, for any $i = 1, 2, \ldots, N$, we have

$$\left\|y_n^i - x_n\right\| \le \left\|y_n^i - x_{n+1}\right\| + \left\|x_{n+1} - x_n\right\| \to 0,$$

and since $\lim_{n\to+\infty} e_n^i = 0$, we obtain that

$$\lim_{n\to+\infty} \left\|y_n^i - (x_n + e_n^i)\right\| = 0. \tag{3.8}$$

Therefore

$$\lim_{n\to+\infty}\left(T_i(x_n+e_n^i)-(x_n+e_n^i)\right)=0. \tag{3.9}$$

Now let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Set $z_n^i=x_n+e_n^i$. Since $x_{n_k}\rightharpoonup v$ and $e_{n_k}^i\to 0$, it is obvious that for any $1\le i\le N$, the sequence $\left\{z_{n_k}^i\right\}_{k\in\mathbb{N}}$ converges weakly to v. We also have $\lim_{k\to+\infty}\left(T_i z_{n_k}^i-z_{n_k}^i\right)=0$ by (3.9). This means that $v\in\hat{F}(T_i)=F(T_i)$. Therefore $v\in F$, as claimed. This proves Step 3.

Step 4. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_F^f(x_0)$ *as* $n\to+\infty$.

From [**26**, Lemma 1.3.1] we know that each $F(T_i)$, $1\le i\le N$, is closed and convex and therefore F is also closed and convex. Thus the assumption that $F\ne\varnothing$ implies that the Bregman projection proj_F^f is well defined.

Let $\tilde{u}=\mathrm{proj}_F^f(x_0)$. Since $x_n=\mathrm{proj}_{Q_n}^f(x_0)$ and F is contained in Q_n, we have $D_f(x_n,x_0)\le D_f(\tilde{u},x_0)$. Therefore Proposition 6 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u}=\mathrm{proj}_F^f(x_0)$, as claimed.

This completes the proof of Theorem 1. □

Remark 1. In a similar way we can also prove that the sequences generated by the following algorithm converge strongly to a common fixed point of finitely many QBFNE operators:

$$\begin{cases} x_0\in X,\\ y_n^i=T_i\left(x_n+e_n^i\right),\\ Q_n^i=\left\{z\in X:\left\langle\nabla f\left(x_n+e_n^i\right)-\nabla f\left(y_n^i\right),z-y_n^i\right\rangle\le 0\right\},\\ Q_n:=\bigcap_{i=1}^N Q_n^i,\\ W_n=\left\{z\in X:\left\langle\nabla f(x_0)-\nabla f(x_n),z-x_n\right\rangle\le 0\right\},\\ x_{n+1}=\mathrm{proj}_{Q_n\cap W_n}^f(x_0), \qquad n=0,1,2,\ldots. \end{cases} \tag{3.10}$$

4. Convex Feasibility Problems

Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. The convex feasibility problem (CFP) is to find an element in the assumed nonempty intersection $\bigcap_{i=1}^{N} K_i$. It is clear that $F\left(\mathrm{proj}^f_{K_i}\right) = K_i$ for any $i = 1, 2, \ldots, N$. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the Bregman projection $\mathrm{proj}^f_{K_i}$ is BFNE, hence QBFNE, and $F\left(\mathrm{proj}^f_{K_i}\right) = \hat{F}\left(\mathrm{proj}^f_{K_i}\right)$ (*cf.* [**26**, Lemma 1.3.2]). Therefore, if we take $T_i = \mathrm{proj}^f_{K_i}$ in Theorem 1, then we get an algorithm for solving convex feasibility problems, which allows for computational errors.

Corollary 1. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1) (with $T_i = \mathrm{proj}^f_{K_i}$). If the sequences of errors $\left\{e^i_n\right\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e^i_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^f_K(x_0)$ as $n \to +\infty$.*

5. Zeroes of Maximal Monotone Mappings

A mapping $A : X \to 2^{X^*}$ is said to be *monotone* if for any $x, y \in \mathrm{dom}\, A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \quad \Longrightarrow \quad \langle \xi - \eta, x - y\rangle \geq 0.$$

(Recall that the set $\mathrm{dom}\, A = \{x \in X \ : \ Ax \neq \varnothing\}$ is called the *effective domain* of such a mapping A.) A monotone mapping A is said to be *maximal* if graph A, the graph of A, is not a proper subset of the graph of any other monotone mapping.

Let $A : X \to 2^{X^*}$ be a maximal monotone mapping. The problem of finding an element $x \in X$ such that $0^* \in Ax$ is very important in Optimization Theory and related fields. In this section we present an algorithm for finding common zeroes of N maximal monotone mappings.

Recall that the *resolvent* of A, denoted by $\mathrm{Res}_A^f : X \to 2^X$, is defined as follows [**5**]:

$$\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

Bauschke, Borwein and Combettes [**5**, Prop. 3.8(iv), p. 604] prove that this resolvent is a single-valued BFNE operator. In addition, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the resolvent Res_A^f is a QBFNE operator (see Section 2.5) which satisfies $F\left(\mathrm{Res}_A^f\right) = \hat{F}\left(\mathrm{Res}_A^f\right)$ (*cf.* [**26**, Lemma 1.3.2]). It is well known that the fixed point set of the resolvent Res_A^f is equal to the set of zeroes of the mapping A, that is, $F\left(\mathrm{Res}_A^f\right) = A^{-1}(0^*)$. If we take $T_i = \mathrm{Res}_{A_i}^f$ in Theorem 1, then we obtain an algorithm for finding common zeroes of finitely many maximal monotone mappings, which allows for computational errors. Note that since each A_i is a maximal monotone mapping, $X^* = \mathrm{ran}\,(\nabla f) = \mathrm{ran}\,(\nabla f + A_i)$ (see [**25**, Proposition 2.3, p. 28] and [**5**, Prop. 3.8(iv), p. 604]) and therefore each T_i is defined on all of X.

Corollary 2. *Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone mappings with $Z := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1) (with $T_i = \mathrm{Res}_{A_i}^f$). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_Z^f(x_0)$ as $n \to +\infty$.*

6. Equilibrium Problems

Let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies the following four conditions [**7**]:

(C1) $g(x, x) = 0$ for all $x \in K$;

(C2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;

(C3) for all $x, y, z \in K$,

$$\limsup_{t \downarrow 0} g\left(tz + (1-t)x, y\right) \leq g(x, y);$$

(C4) for each $x \in K$, $g(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problem corresponding to g is to find $\bar{x} \in K$ such that

$$g(\bar{x}, y) \geq 0 \quad \forall y \in K. \tag{6.1}$$

The set of solutions of (6.1) is denoted by $EP(g)$.

The *resolvent of a bifunction* $g : K \times K \to \mathbb{R}$ [**17**] is the operator $\mathrm{Res}_g^f : X \to 2^K$, defined by

$$\mathrm{Res}_g^f(x) = \{z \in K : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in K\}.$$

In the following proposition we list several properties of these resolvents. Recall that f is said to be *coercive* if $\lim_{\|x\| \to +\infty} (f(x) / \|x\|) = +\infty$.

Proposition 7 (*cf.* [**27**, Lemmas 1 and 2, pp. 130–131])**.** *Let* $f : X \to (-\infty, +\infty)$ *be a coercive Legendre function. Let* K *be a closed and convex subset of* X*. If the bifunction* $g : K \times K \to \mathbb{R}$ *satisfies conditions* (C1)–(C4), *then*

(i) $\mathrm{dom}\,(\mathrm{Res}_g^f) = X$*;*

(ii) Res_g^f *is single-valued;*

(iii) Res_g^f *is a BFNE operator;*

(iv) the set of fixed points of Res_g^f *is the solution set of the corresponding equilibrium problem, i.e.,* $F(\mathrm{Res}_g^f) = EP(g)$;

(iv) $EP(g)$ *is a closed and convex subset of* K;

(vi) For all $x \in X$ *and for all* $u \in F(\mathrm{Res}_g^f)$*, we have*

$$D_f\left(u, \mathrm{Res}_g^f(x)\right) + D_f\left(\mathrm{Res}_g^f(x), x\right) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the resolvent Res_g^f is single-valued (Proposition

7(ii)), QBFNE (see Section 2.5 and Proposition 7(vi)) and satisfies $F\left(\mathrm{Res}_g^f\right) = \hat{F}\left(\mathrm{Res}_g^f\right)$ (*cf.* [**26**, Lemma 1.3.2]). From Proposition 7(iv) we also know that $F\left(\mathrm{Res}_g^f\right) = EP\,(g)$. So, if we take $T_i = \mathrm{Res}_{g_i}^f$ in Theorem 1, then we get an algorithm for finding common solutions to the equilibrium problems corresponding to finitely many bifunctions, which allows for computational errors. Note that each T_i is defined on all of X (Proposition 7(i)).

Corollary 3. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) such that $E := \bigcap_{i=1}^N EP\,(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1) (with $T_i = \mathrm{Res}_{g_i}^f$). If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfy $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_E^f(x_0)$ as $n \to +\infty$.*

7. Zeroes of Bregman Inverse Strongly Monotone Mappings

Using our method, we can find common zeroes for another class of mappings, namely, Bregman inverse strongly monotone mappings. This class of mappings was introduced by Butnariu and Kassay (see [**14**]). Let $A : X \to 2^{X^*}$ be a mapping. We assume that the Legendre function f satisfies the following range condition:

$$\mathrm{ran}\,(\nabla f - A) \subseteq \mathrm{ran}\,(\nabla f)\,. \tag{7.1}$$

The mapping $A : X \to 2^{X^*}$ is called *Bregman inverse strongly monotone* (BISM for short) if

$$(\mathrm{dom}\,A)\bigcap(\mathrm{int\,dom}f) \neq \varnothing$$

and for any $x, y \in \mathrm{int\,dom}f$, and each $\xi \in Ax$, $\eta \in Ay$, we have

$$\langle \xi - \eta, \nabla f^*\left(\nabla f\,(x) - \xi\right) - \nabla f^*\left(\nabla f\,(y) - \eta\right)\rangle \geq 0.$$

For any mapping $A : X \to 2^{X^*}$, the *anti-resolvent* $A^f : X \to 2^X$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that $\operatorname{dom} A^f \subseteq (\operatorname{dom} A) \cap (\operatorname{int} \operatorname{dom} f)$ and $\operatorname{ran} A^f \subseteq \operatorname{int} \operatorname{dom} f$.

It is known (see [**14**, Lemma 3.5 (c) and (d), p. 2109]) that the mapping A is BISM if and only if the anti-resolvent A^f is a (single-valued) BFNE operator. For examples of BISM mappings and more information on this new class of mappings see [**14**]. Before presenting consequences of our main results, we note several properties of this class of mappings and of the anti-resolvent.

From the definition of the anti-resolvent and [**14**, Lemma 3.5, p. 2109] we obtain the following proposition.

Proposition 8. *Let $f : X \to (-\infty, +\infty]$ be a Legendre function which satisfies the range condition (7.1) and let $A : X \to 2^{X^*}$ be a BISM mapping such that $A^{-1}(0^*) \neq \varnothing$. Then the following statements hold:*

(i) $A^{-1}(0^*) = F(A^f)$;

(ii) for any $u \in A^{-1}(0^*)$ and $x \in \operatorname{dom} A^f$, we have

$$D_f\left(u, A^f x\right) + D_f\left(A^f x, x\right) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then the anti-resolvent A^f is a single-valued QBFNE operator (see Section 2.5) which satisfies $F\left(A^f\right) = \hat{F}\left(A^f\right)$ (*cf.* [**26**, Lemma 1.3.2]).

Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X and let $T_i : K_i \to K_i$ for each $i = 1, 2, \ldots, N$. Assume that $\bigcap_{i=1}^N K_i \neq \varnothing$ and consider

Algorithm (3.1) without computational errors:

$$
\begin{cases}
x_0 \in K = \bigcap_{i=1}^{N} K_i, \\
Q_0^i = K_i, \qquad i = 1, 2, \ldots, N, \\
y_n^i = T_i(x_n), \\
Q_{n+1}^i = \left\{ z \in Q_n^i : \left\langle \nabla f(x_n) - \nabla f(y_n^i), z - y_n^i \right\rangle \leq 0 \right\}, \\
Q_{n+1} := \bigcap_{i=1}^{N} Q_{n+1}^i, \\
x_{n+1} = \mathrm{proj}_{Q_{n+1}}^f (x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases} \tag{7.2}
$$

In the next result we assume that each one of the operators A_i satisfies $K_i \subset \operatorname{dom} A_i$ and that $f : X \to \mathbb{R}$. From the range condition (7.1) we get that $\operatorname{dom} A_i^f = (\operatorname{dom} A_i) \cap (\operatorname{int} \operatorname{dom} f) = \operatorname{dom} A_i$ because in our case $\operatorname{int} \operatorname{dom} f = X$. From Proposition 8(i) we know that $F\left(A_i^f\right) = A_i^{-1}(0^*)$. So, if we take $T_i = A_i^f$ in Theorem 1, then we get an algorithm for finding common zeroes of finitely many BISM mappings.

Corollary 4. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, be N BISM mappings such that $K_i \subset \operatorname{dom} A_i$ and $Z := \bigcap_{i=1}^{N} A_i^{-1}(0^*) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Assume that the range condition (7.1) is satisfied for each A_i. Then, for each $x_0 \in K$, there are sequences $\{x_n\}_{n \in \mathbb{N}}$ which satisfy (7.2) (with $T_i = A_i^f$) and each such sequence $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to* $\mathrm{proj}_Z^f(x_0)$ *as $n \to +\infty$.*

Remark 2. Note that in this section we can still allow for possible computational errors if we assume that there exists $\varepsilon > 0$ such that $\left\| e_n^i \right\| < \varepsilon$ for each $i = 1, 2, \ldots, N$ and for all $n \in \mathbb{N}$, and that the relevant operators are defined not only on K, but also on $K_\varepsilon := \{x \in X : d(x, K) < \varepsilon\}$, where $d(x, K) := \inf \{\|x - y\| : y \in K\}$.

8. Mixed Problems

There are many papers which propose algorithms for finding common solutions to mixed problems; for example, common solutions to two fixed point problems and, say, an equilibrium problem. If we combine Sections 4–6, then we can find common solutions to any finite number of problems such as fixed point problems, CFP, finding zeroes of maximal monotone mappings and EP.

For instance, if we wish to find a common solution to two fixed point problems and an equilibrium problem, then we define the operators in Theorem 1 as follows: $T_1 = T$, $T_2 = S$ and $T_3 = \mathrm{Res}_g^f$, where T and S are QBFNE operators which satisfy $F(T_i) = \hat{F}(T_i)$, $i = 1, 2$, and $g : C \times C \to \mathbb{R}$ is a bifunction that satisfies conditions (C1)–(C4). If

$$F = F(T) \bigcap F(S) \bigcap EP(g) \neq \varnothing,$$

then Algorithm (3.1) generates sequences $\{x_n\}_{n \in \mathbb{N}}$ which converge strongly to $\mathrm{proj}_F^f(x_0)$ as $n \to +\infty$.

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