Three Strong Convergence Theorems Regarding Iterative Methods for Solving Equilibrium Problems in Reflexive Banach Spaces

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Abstract. We establish three strong convergence theorems regarding iterative methods for finding a common solution to the equilibrium problems corresponding to finitely many bifunctions in reflexive Banach spaces. In all three theorems we also take into account possible computational errors.

1. Introduction

Let X denote a real reflexive Banach space with norm $\|\cdot\|$ and let X^* stand for the (topological) dual of X endowed with the induced norm $\|\cdot\|_*$. We denote the value of the functional $\xi \in X^*$ at $x \in X$ by $\langle \xi, x \rangle$. In this paper $f : X \to (-\infty, +\infty]$ is always a proper, lower semicontinuous and convex function, and $f^* : X^* \to (-\infty, +\infty]$ is the Fenchel conjugate of f. The set of nonnegative integers is denoted by $\mathbb{N}$.

Let K be a closed and convex subset of X and let $g : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem corresponding to g is to find $\bar{x} \in K$ such that

$$g(\bar{x}, y) \geq 0 \quad \forall y \in K. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $EP(g)$. This problem contains as special cases many optimization, fixed point and variational inequality problems (see [**7**] and [**15**] for more details). In 2005 Combettes and Hirstoaga [**15**] introduced an iterative scheme in Hilbert space for finding the best approximation to the initial datum from $EP(g)$ when $EP(g)$ is nonempty, and established a strong convergence theorem for their scheme. More recently, Takahashi and Zembayashi [**25**] have proposed an algorithm for solving equilibrium problems in those Banach spaces X which are both uniformly convex and uniformly smooth. More algorithms can be found, for example, in [**24**]. In the present paper we propose three algorithms (see Algorithms (3.1), (4.1) and (5.1) below) for solving (common) equilibrium problems in general reflexive Banach spaces using a well chosen convex function f, as well as the Bregman distance and projection associated with it (see Section 2.3). Our algorithms are more flexible than those previously used because they leave us the freedom of fitting the function f to the nature of the bifunctions g and of the space X. If X is a uniformly convex and uniformly smooth Banach space, then we can choose $f(x) = (1/2)\|x\|^2$ in our algorithms. However, this choice may make the computations quite difficult in some Banach spaces. These computations can be simplified by an appropriate choice of f. For instance, if $X = \ell^p$ or $X = L^p$ with $p \in (1, \infty)$, then we may choose $f(x) = (1/p)\|x\|^p$. All three of our algorithms allow for certain computational errors. These algorithms are similar to, but different from those we have recently studied in [**17**, Theorem 4.2, p. 35] and [**19**, Corollaries 5 and 6], where the algorithms approximate common zeroes of finitely many maximal monotone operators. Our main results (Theorems 1, 2 and 3) are formulated and proved in Sections 3, 4 and 5, respectively. Their proofs, although similar, differ from each other in significant details. Each one of these sections also contains three corollaries which are deduced from the theorem established in that section. The next section is devoted to several preliminary definitions and results.

From now on we denote the set $\{x \in X : f(x) < +\infty\}$ by $\operatorname{dom} f$ and the set $\{f(x) : x \in \operatorname{dom} f\}$ by $\operatorname{ran} f$. The interior of a set K is denoted by $\operatorname{int} K$.

2. Preliminaries

2.1. Some facts about Legendre functions. Legendre functions mapping a general Banach space X into $(-\infty,+\infty]$ are defined in [**3**]. According to [**3**, Theorems 5.4 and 5.6], since X is reflexive, the function f is *Legendre* if and only if it satisfies the following two conditions:

(L1) The interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is Gâteaux differentiable (see below) on $\operatorname{int}\operatorname{dom} f$, and

$$\operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f;$$

(L2) The interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f^*$, and

$$\operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*.$$

Since X is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [**8**, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},$$

$$\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$$

and

$$\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f.$$

Also, conditions (L1) and (L2), in conjunction with [**3**, Theorem 5.4], imply that the functions f and f^* are strictly convex on the interior of their respective domains.

Several interesting examples of Legendre functions are presented in [**2**] and [**3**]. Among them are the functions $(1/s)\left\|\cdot\right\|^s$ with $s \in (1,\infty)$, where the Banach space X is smooth and strictly convex and, in particular, a Hilbert space. From now on we assume that the convex function $f : X \to (-\infty,+\infty]$ is Legendre.

2.2. A property of gradients. For any $x \in \operatorname{int} \operatorname{dom} f$ and $y \in X$, we denote by $f^{\circ}(x,y)$ the *right-hand derivative of* f *at* x in the direction y, that is,

$$f^{\circ}(x,y) := \lim_{t \searrow 0} \frac{f(x+ty) - f(x)}{t}.$$

The function f is called *Gâteaux differentiable at* x if $\lim_{t \to 0} \left(f(x+ty) - f(x)\right)/t$ exists for any y. In this case $f^{\circ}(x,y)$ coincides with $\left(\nabla f\right)(x)$, the value of the *gradient* ∇f of f at x. The function f is said to be *Fréchet differentiable at* x if this limit is attained uniformly for $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable on a subset* E *of* X if the limit is attained uniformly for $x \in E$ and $\|y\| = 1$. We will need the following result.

Proposition 1 (*cf.* [**16**, Proposition 2.1, p. 474]). *If* $f : X \to \mathbb{R}$ *is uniformly Fréchet differentiable and bounded on bounded subsets of* X*, then* ∇f *is uniformly continuous on bounded subsets of* X *from the strong topology of* X *to the strong topology of* X^{*}.

2.3. Some facts about totally convex functions. Let $f : X \to (-\infty, +\infty]$ be a convex function which is Gâteaux differentiable in $\operatorname{int} \operatorname{dom} f$. The function $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$, defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \,, \tag{2.1}$$

is called the *Bregman distance with respect to* f (*cf.* [**14**]). The Bregman distance has the following two important properties, called the *three point identity*: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int} \operatorname{dom} f$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle \,, \tag{2.2}$$

and the *four point identity*: for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int} \operatorname{dom} f$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle \,. \tag{2.3}$$

Recall that, according to [**11**, Section 1.2, p. 17] (see also [**10**]), the function f is called *totally convex at a point* $x \in \operatorname{int} \operatorname{dom} f$ if its *modulus of total convexity at*

x, that is, the function $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x,t) := \inf \{D_f(y,x) : y \in \operatorname{dom}\, f,\ \ \|y - x\| = t\},$$

is positive whenever $t > 0$. The function f is called *totally convex* when it is totally convex at every point $x \in \operatorname{int}\operatorname{dom}\, f$. In addition, the function f is called *totally convex on bounded sets* if $\upsilon_f(E,t)$ is positive for any nonempty and bounded subset E of X and for any $t > 0$, where the *modulus of total convexity of the function* f on the set E is the function $\upsilon_f : \operatorname{int}\operatorname{dom}\, f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\upsilon_f(E,t) := \inf \{\upsilon_f(x,t) : x \in E \cap \operatorname{int}\operatorname{dom}\, f\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [**13**, Theorem 2.10, p. 9]).

Examples of totally convex functions can be found, for instance, in [**11, 13**]. The next proposition turns out to be very useful in the proof of Theorems 1, 2 and 3 below.

Proposition 2 (*cf.* [**21**, Proposition 2.2, p. 3]). *If* $x \in \operatorname{int}\operatorname{dom}\, f$, *then the following statements are equivalent:*

(i) The function f *is totally convex at* x*;*

(ii) For any sequence $\{y_n\}_{n\in\mathbb{N}} \subset \operatorname{dom}\, f$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \ \Rightarrow\ \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Recall that the function f is called *sequentially consistent* (see [**13**]) if for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{int}\operatorname{dom}\, f$ and $\operatorname{dom}\, f$, respectively, such that the first one is bounded,

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \ \Rightarrow\ \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Proposition 3 (*cf.* [**11**, Lemma 2.1.2, p. 67]). *The function* f *is totally convex on bounded sets if and only if it is sequentially consistent.*

Recall that the *Bregman projection* (*cf.* [**9**]) of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty, closed and convex set $K \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_K^f(x) \in K$ satisfying

$$D_f\left(\operatorname{proj}_K^f(x), x\right) = \inf\left\{D_f\left(y, x\right) : y \in K\right\}.$$

Similarly to the nearest point projection in Hilbert spaces, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Proposition 4 (*cf.* [**13**, Corollary 4.4, p. 23]). *Suppose that f is Gâteaux differentiable and totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following conditions are equivalent*:

(i) The vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\left\langle \nabla f\left(x\right) - \nabla f\left(z\right), z - y\right\rangle \geq 0 \qquad \forall y \in K;$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f\left(y, z\right) + D_f\left(z, x\right) \leq D_f\left(y, x\right) \qquad \forall y \in K.$$

The following two propositions exhibit two additional properties of totally convex functions.

Proposition 5 (*cf.* [**17**, Lemma 3.1, p. 31]). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function. If $x_0 \in X$ and the sequence $\left\{D_f(x_n, x_0)\right\}_{n\in\mathbb{N}}$ is bounded, then the sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ is bounded too.*

Proposition 6 (*cf.* [**17**, Lemma 3.2, p. 31]). *Let $f : X \to \mathbb{R}$ be a Legendre and totally convex function, $x_0 \in X$, and let K be a nonempty, closed and convex subset of X. Suppose that the sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ is bounded and any weak subsequential limit of $\left\{x_n\right\}_{n\in\mathbb{N}}$ belongs to K. If $D_f\left(x_n, x_0\right) \leq D_f\left(\operatorname{proj}_K^f(x_0), x_0\right)$ for any $n \in \mathbb{N}$, then $\left\{x_n\right\}_{n\in\mathbb{N}}$ converges strongly to* $\operatorname{proj}_K^f(x_0)$.

2.4. Some facts about the resolvent of a bifunction. Let K be a closed and convex subset of X, and let $g : K \times K \to \mathbb{R}$ be a bifunction satisfying the following conditions [**7, 15**]:

(C1) $g(x,x) = 0$ for all $x \in K$;

(C2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \le 0$ for all $x, y \in K$;

(C3) for all $x, y, z \in K$,

$$\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \le g(x,y);$$

(C4) for each $x \in K$, $g(x, \cdot)$ is convex and lower semicontinuous.

Let λ be a positive real number. The *resolvent of a bifunction* $g : K \times K \to \mathbb{R}$ [**15**] is the mapping $\mathrm{Res}^f_{\lambda g} : X \to 2^K$, defined by

$$\mathrm{Res}^f_{\lambda g}(x) = \{z \in K : \lambda g(z,y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \ge 0 \quad \forall y \in K\}.$$

Recall that the function f is said to be *coercive* if $\lim_{\|x\| \to +\infty} (f(x) / \|x\|) = +\infty$. If K is a subset of $\operatorname{int}\operatorname{dom} f$, then the operator $T : K \to K$ is called *Bregman firmly nonexpansive (BFNE* for short*)* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

for all $x, y \in K$. See [**4, 18**] for more information on BFNE operators.

Now we list some properties of the resolvent of a bifunction.

Proposition 7 (*cf.* [**19**, Lemmas 1 and 2. pp. 130-131]). *Let* $f : X \to (-\infty, +\infty]$ *be a coercive Legendre function. Let* K *be a closed and convex subset of* X. *If the bifunction* $g : K \times K \to \mathbb{R}$ *satisfies conditions* (C1)–(C4), *then*:

(i) $\mathrm{dom}(\mathrm{Res}^f_g) = X$;

(ii) Res^f_g *is single-valued;*

(iii) Res^f_g *is a BFNE operator;*

(iv) the set of fixed points of Res^f_g *is the solution set of the corresponding equilibrium problem, i.e.,* $F(\mathrm{Res}^f_g) = EP(g)$;

(v) $EP(g)$ *is a closed and convex subset of* K;

(vi) for all $x \in X$ and for all $u \in F(\mathrm{Res}_g^f)$, we have

$$D_f\left(u, \mathrm{Res}_g^f(x)\right) + D_f\left(\mathrm{Res}_g^f(x), x\right) \leq D_f(u, x).$$

3. Algorithm I

In this section we present an algorithm which is motivated by the algorithm proposed by Bauschke and Combettes [**5**] (see also Solodov and Svaiter [**22**]). More precisely, we study the following algorithm when $E := \bigcap_{i=1}^{N} EP(g_i) \neq \varnothing$:

$$\text{(3.1)} \qquad \begin{cases} x_0 \in X, \\ y_n^i = \mathrm{Res}_{\lambda_n^i g_i}^f(x_n + e_n^i), \\ C_n^i = \left\{z \in X : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\}, \\ C_n := \bigcap_{i=1}^{N} C_n^i, \\ Q_n = \left\{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\right\}, \\ x_{n+1} = \mathrm{proj}_{C_n \cap Q_n}^f(x_0), \qquad n = 0, 1, 2, \ldots. \end{cases}$$

Theorem 1. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^{N} EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1). If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to+\infty} \lambda_n^i > 0$, and the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_E^f(x_0)$ as $n \to +\infty$.*

Proof. We divide our proof into four steps.

Step 1. *There are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1).*

From Proposition 7(i) we know that $\mathrm{dom}\, \mathrm{Res}_{\lambda_n^i g_i}^f = X$ for any $i = 1, 2, \ldots, N$. Therefore each y_n^i is well-defined whenever x_n is. Let $n \in \mathbb{N}$. It is not difficult to check that the sets C_n^i are closed and convex for each $i = 1, 2, \ldots, N$. Hence their intersection C_n is also closed and convex. It is also obvious that Q_n is a closed and convex set. Let $u \in E$. For any $n \in \mathbb{N}$, we obtain from Proposition 7(vi) that

$$D_f\left(u, y_n^i\right) = D_f\left(u, \operatorname{Res}^f_{\lambda_n^i g_i}(x_n + e_n^i)\right) \leq D_f\left(u, x_n + e_n^i\right),$$

which implies that $u \in C_n^i$. Since this holds for any $i = 1, 2, \ldots, N$, it follows that $u \in C_n$. Thus $E \subset C_n$ for any $n \in \mathbb{N}$. On the other hand, it is obvious that $E \subset Q_0 = X$. Thus $E \subset C_0 \bigcap Q_0$, and therefore $x_1 = \operatorname{proj}^f_{C_0 \cap Q_0}(x_0)$ is well defined. Now suppose that $E \subset C_{n-1} \bigcap Q_{n-1}$ for some $n \geq 1$. Then $x_n = \operatorname{proj}^f_{C_{n-1} \cap Q_{n-1}}(x_0)$ is well defined because $C_{n-1} \bigcap Q_{n-1}$ is a nonempty, closed and convex subset of X. So from Proposition 4(ii) we have

$$\langle \nabla f(x_0) - \nabla f(x_n), y - x_n \rangle \leq 0,$$

for any $y \in C_{n-1} \bigcap Q_{n-1}$. Hence we obtain that $E \subset Q_n$. Therefore $E \subset C_n \bigcap Q_n$ and hence $x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n}(x_0)$ is well defined. Consequently, we see that $E \subset C_n \bigcap Q_n$ for any $n \in \mathbb{N}$. Thus the sequence we constructed is indeed well defined and satisfies (3.1), as claimed.

From now on we fix an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ which satisfies (3.1).

Step 2. *The sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded.*

It follows from the definition of the set Q_n and Proposition 4(ii) that $\operatorname{proj}^f_{Q_n}(x_0) = x_n$. Furthermore, by Proposition 4(iii), for each $u \in E$, we have

$$\begin{aligned} D_f\left(x_n, x_0\right) &= D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \\ &\leq D_f\left(u, x_0\right) - D_f\left(u, \operatorname{proj}^f_{Q_n}(x_0)\right) \leq D_f\left(u, x_0\right). \end{aligned}$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in E$. Therefore by Proposition 5 the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded too, as claimed.

Step 3. *Every weak subsequential limit of $\{x_n\}_{n \in \mathbb{N}}$ belongs to E.*

It follows from the definition of Q_n and Proposition 4(ii) that $\operatorname{proj}^f_{Q_n}(x_0) = x_n$. Since $x_{n+1} \in Q_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1}, \operatorname{proj}^f_{Q_n}(x_0)\right) + D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and hence

$$D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_0\right) \leq D_f\left(x_{n+1}, x_0\right). \tag{3.2}$$

Therefore the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Step 2), $\lim_{n\to+\infty} D_f(x_n, x_0)$ exists. Thus from (3.2) it follows that

$$\lim_{n\to+\infty} D_f\left(x_{n+1}, x_n\right) = 0. \tag{3.3}$$

Proposition 3 now implies that $\lim_{n\to+\infty}\left(x_{n+1} - x_n\right) = 0$. For any $i = 1, 2, \ldots, N$, it follows from the definition of the Bregman distance (see (2.1)) that

$$D_f\left(x_n, x_n + e_n^i\right) = f\left(x_n\right) - f\left(x_n + e_n^i\right) - \left\langle \nabla f(x_n + e_n^i), x_n - \left(x_n + e_n^i\right)\right\rangle =$$

$$f\left(x_n\right) - f\left(x_n + e_n^i\right) + \left\langle \nabla f(x_n + e_n^i), e_n^i\right\rangle.$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [**11**, Proposition 1.1.11, p. 17]). In addition, f is uniformly Fréchet differentiable and therefore it is uniformly continuous on bounded subsets (see [**1**, Theorem 1.8, p. 13]). Hence, since $\lim_{n\to+\infty} e_n^i = 0$, it follows that

$$\lim_{n\to+\infty} D_f\left(x_n, x_n + e_n^i\right) = 0. \tag{3.4}$$

For any $i = 1, 2, \ldots, N$, it follows from the three point identity (see (2.2)) that

$$D_f\left(x_{n+1}, x_n + e_n^i\right) = D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_n + e_n^i\right)$$

$$+ \left\langle \nabla f(x_n) - \nabla f(x_n + e_n^i), x_{n+1} - x_n\right\rangle.$$

Since $\lim_{n\to+\infty}\left(x_{n+1} - x_n\right) = 0$ and ∇f is bounded on bounded subsets of X, (3.3) and (3.4) imply that

$$\lim_{n\to+\infty} D_f\left(x_{n+1}, x_n + e_n^i\right) = 0.$$

For any $i = 1, 2, \ldots, N$, it follows from the inclusion $x_{n+1} \in C_n^i$ that

$$D_f\left(x_{n+1}, y_n^i\right) \leq D_f\left(x_{n+1}, x_n + e_n^i\right).$$

Hence $\lim_{n\to+\infty} D_f\left(x_{n+1}, y_n^i\right) = 0$. Proposition 3 now implies that $\lim_{n\to+\infty}\left(y_n^i - x_{n+1}\right) = 0$. Therefore, for any $i = 1, 2, \ldots, N$, we have

$$\left\|y_n^i - x_n\right\| \leq \left\|y_n^i - x_{n+1}\right\| + \left\|x_{n+1} - x_n\right\| \to 0.$$

This means that the sequence $\left\{y_n^i\right\}_{n\in\mathbb{N}}$ is bounded for any $i = 1, 2, \ldots, N$. Since f is uniformly Fréchet differentiable, it follows from Proposition 1 that

$$\lim_{n\to+\infty}\left\|\nabla f\left(y_n^i\right) - \nabla f\left(x_n\right)\right\|_* = 0,$$

and since $\lim_{n\to+\infty} e_n^i = 0$, it also follows that

$$\lim_{n\to+\infty}\left\|\nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right)\right\|_* = 0 \tag{3.5}$$

for any $i = 1, 2, \ldots, N$. By the definition of y_n^i we know that

$$\lambda_n^i g_i\left(y_n^i, y\right) + \left\langle\nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right), y - y_n^i\right\rangle \geq 0$$

for all $y \in K_i$. Hence from condition (C2) it follows that

$$\left\langle\nabla f\left(y_n^i\right) - \nabla f\left(x_n + e_n^i\right), y - y_n^i\right\rangle \geq -\lambda_n^i g_i\left(y_n^i, y\right) \geq \lambda_n^i g_i\left(y, y_n^i\right) \tag{3.6}$$

for all $y \in K_i$. Now let $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\left\{x_n\right\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Then $\left\{y_{n_k}^i\right\}_{k\in\mathbb{N}}$ also converges weakly to v for any $i = 1, 2, \ldots, N$. Replacing n by n_k in (3.6), we get that

$$\left\langle\nabla f\left(y_{n_k}^i\right) - \nabla f\left(x_{n_k} + e_{n_k}^i\right), y - y_{n_k}^i\right\rangle \geq \lambda_{n_k}^i g\left(y, y_{n_k}^i\right). \tag{3.7}$$

Since the sequence $\left\{y_{n_k}^i\right\}_{k\in\mathbb{N}}$ is bounded, condition (C4) holds, and $\liminf_{k\to+\infty}\lambda_{n_k}^i > 0$, it follows from (3.5) and (3.7) that

$$g_i\left(y,v\right)\leq 0, \tag{3.8}$$

for each $y\in K_i$ and for any $i=1,2,\ldots,N$. For any $t\in(0,1]$, we now define $y_t=ty+(1-t)\,v$. Let $i=1,2,\ldots,N$, since y and v belongs to K_i, it follows from the convexity of K_i that $y_t\in K_i$ too. Hence $g_i\left(y_t,v\right)\leq 0$ for any $i=1,2,\ldots,N$. So, from conditions (C1) and (C4), and (3.8) it follows that

$$0=g_i\left(y_t,y_t\right)\leq tg_i\left(y_t,y\right)+(1-t)\,g_i\left(y_t,v\right)\leq tg_i\left(y_t,y\right).$$

Dividing by t, we obtain that $g_i\left(y_t,y\right)\geq 0$ for all $y\in K_i$. Letting $t\downarrow 0$, and using condition (C3), we see that $g_i\left(v,y\right)\geq 0$ for all $y\in K_i$. Thus $v\in EP\left(g_i\right)$ for any $i=1,2,\ldots,N$. Therefore $v\in E$ and this proves Step 3.

Step 4. *The sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *converges strongly to* $\operatorname{proj}_E^f(x_0)$ *as* $n\to+\infty$.

From Proposition 7(v) it follows that $EP\left(g_i\right)$ is closed and convex for any $i=1,2,\ldots,N$. Therefore E is nonempty, closed and convex, and the Bregman projection proj_E^f is well defined. Let $\tilde{u}=\operatorname{proj}_E^f(x_0)$. Since $x_{n+1}=\operatorname{proj}_{C_n\cap Q_n}^f(x_0)$ and E is contained in $C_n\cap Q_n$, we have $D_f\left(x_{n+1},x_0\right)\leq D_f\left(\tilde{u},x_0\right)$. Therefore Proposition 6 implies that $\left\{x_n\right\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u}=\operatorname{proj}_E^f(x_0)$, as claimed.

This completes the proof of Theorem 1. □

Now we present three consequences of Theorem 1. First we study the following algorithm:

$$\begin{cases} x_0\in X, \\ y_n=\operatorname{Res}_{\lambda_n g}^f(x_n), \\ C_n=\left\{z\in X: D_f\left(z,y_n\right)\leq D_f\left(z,x_n\right)\right\}, \\ Q_n=\left\{z\in X:\left\langle\nabla f(x_0)-\nabla f(x_n),z-x_n\right\rangle\leq 0\right\}, \\ x_{n+1}=\operatorname{proj}_{C_n\cap Q_n}^f(x_0), \qquad n=0,1,2,\ldots. \end{cases} \tag{3.9}$$

Algorithm (3.9) is a special case of Algorithm (3.1) when $e_n = 0$ for all $n \in \mathbb{N}$ and $N = 1$. Therefore we obtain the following result as a direct consequence of Theorem 1.

Corollary 1. *Let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies conditions (C1)–(C4) with $EP(g) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X, and suppose that $\liminf_{n\to+\infty} \lambda_n > 0$. Then, for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (3.9) converges strongly to $\operatorname{proj}^f_{EP(g)}(x_0)$ as $n \to +\infty$.*

The following corollary [**19**, Corollary 5] follows immediately from Theorem 1 when we take $\lambda^i_n = 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$.

Corollary 2. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (3.1). If, for each $i = 1, 2, \ldots, N$, the sequence of errors $\{e^i_n\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to+\infty} e^i_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}^f_E(x_0)$ as $n \to +\infty$.*

A notable corollary of Theorem 1 occurs when the space X is both uniformly smooth and uniformly convex. In this case the function $f(x) = (1/2)\|x\|^2$ is coercive and Legendre (*cf.* [**3**, Lemma 6.2, p. 24]), and uniformly Fréchet differentiable on bounded subsets of X. According to [**12**, Corollary 1(ii), p. 325], f is sequentially consistent (because X is uniformly convex) and hence f is totally convex on bounded subsets of X (see Proposition 3). Therefore Theorem 1 holds in this setting and leads to the following result, which is a special case of Theorem 3.1 in

[**25**]. More precisely, we consider the following algorithm:

$$(3.10)\qquad \begin{cases} x_0 \in X, \\ y_n = \operatorname{Res}^f_{\lambda_n g}(x_n), \\ C_n = \left\{ z \in X : \phi\left(z, y_n\right) \leq \phi\left(z, x_n\right) \right\}, \\ Q_n = \left\{ z \in X : \langle J(x_0) - J(x_n), z - x_n \rangle \leq 0 \right\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $J : X \to X^*$ is the normalized duality mapping of the space X, $\phi\left(y, x\right) = \left\| y \right\|^2 - 2 \left\langle Jx, y \right\rangle + \left\| x \right\|^2$ and P_K is the Bregman projection onto K with respect to $f(x) = (1/2) \left\| x \right\|^2$.

Corollary 3. *Let X be a uniformly smooth and uniformly convex Banach space, and let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies conditions (C1)–(C4) with $EP\left(g\right) \neq \varnothing$. If $\liminf_{n \to +\infty} \lambda_n > 0$, then for each $x_0 \in X$, the sequence $\left\{x_n\right\}_{n \in \mathbb{N}}$ generated by (3.10) converges strongly to $P_{EP(g)}(x_0)$ as $n \to +\infty$.*

4. Algorithm II

In this section we present a result which is similar to Theorem 1, but with a different construction of the sequence $\left\{x_n\right\}_{n \in \mathbb{N}}$. The following algorithm is based on the concept of the so-called shrinking projection method, which was introduced by Takahashi, Takeuchi and Kubota in [**23**]. More precisely, we study the following algorithm when $E := \bigcap_{i=1}^{N} EP\left(g_i\right) \neq \varnothing$:

$$
\begin{cases}
x_0 \in X, \\
C_0^i = X, \qquad i = 1, 2, \ldots, N, \\
y_n^i = \mathrm{Res}^f_{\lambda_n^i g_i}(x_n + e_n^i), \\
C_{n+1}^i = \left\{ z \in C_n^i : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right) \right\}, \\
C_{n+1} := \bigcap_{i=1}^N C_{n+1}^i, \\
x_{n+1} = \mathrm{proj}^f_{C_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases} \tag{4.1}
$$

Theorem 2. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP\left(g_i\right) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\left\{x_n\right\}_{n\in\mathbb{N}}$ which satisfy (4.1). If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to+\infty} \lambda_n^i > 0$, and the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^f_E(x_0)$ as $n \to +\infty$.*

Proof. Again we divide our proof into four steps.

Step 1. *There are sequences $\left\{x_n\right\}_{n\in\mathbb{N}}$ which satisfy (4.1).*

From Proposition 7(i) we know that $\mathrm{dom}\,\mathrm{Res}^f_{\lambda_n^i g_i} = X$ for any $i = 1, 2, \ldots, N$. Therefore each y_n^i is well-defined whenever x_n is. Let $n \in \mathbb{N}$. It is not difficult to check that the sets C_n^i are closed and convex for any $i = 1, 2, \ldots, N$. Hence their intersection C_n is also closed and convex. Let $u \in E$. For any $n \in \mathbb{N}$, we obtain from Proposition 7(vi) that

$$
D_f\left(u, y_n^i\right) = D_f\left(u, \mathrm{Res}^f_{\lambda_n^i g_i}(x_n + e_n^i)\right) \leq D_f\left(u, x_n + e_n^i\right),
$$

which implies that $u \in C_{n+1}^i$. Since this holds for any $i = 1, 2, \ldots, N$, it follows that $u \in C_{n+1}$. Thus $E \subset C_n$ for any $n \in \mathbb{N}$.

From now on we fix an arbitrary sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ satisfying (4.1).

Step 2. *The sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ is bounded.*

It follows from Proposition 4(iii) that, for each $u \in E$, we have

$$D_f\left(x_n, x_0\right) = D_f\left(\operatorname{proj}^f_{C_n}(x_0), x_0\right) \leq D_f\left(u, x_0\right) - D_f\left(u, \operatorname{proj}^f_{C_n}(x_0)\right) \leq D_f\left(u, x_0\right).$$

Hence the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is bounded by $D_f(u, x_0)$ for any $u \in E$. Therefore by Proposition 5 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Step 3. *Every weak subsequential limit of* $\{x_n\}_{n\in\mathbb{N}}$ *belongs to* E.

Since $x_{n+1} \in C_{n+1} \subset C_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1}, \operatorname{proj}^f_{C_n}(x_0)\right) + D_f\left(\operatorname{proj}^f_{C_n}(x_0), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and hence

$$D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_0\right) \leq D_f\left(x_{n+1}, x_0\right). \tag{4.2}$$

Therefore the sequence $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Step 2), $\lim_{n\to+\infty} D_f(x_n, x_0)$ exists. Thus from (4.2) it follows that

$$\lim_{n\to+\infty} D_f\left(x_{n+1}, x_n\right) = 0.$$

Now, using an argument similar to the one we employed in the proof of Theorem 1 (see Step 3 there), we get the conclusion of Step 3.

Step 4. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\operatorname{proj}^f_E(x_0)$ *as* $n \to +\infty$.

From Proposition 7(v) it follows that $EP\left(g_i\right)$ is closed and convex for any $i = 1, 2, \ldots, N$. Therefore E is nonempty, closed and convex, and the Bregman projection proj^f_E is well defined. Let $\tilde{u} = \operatorname{proj}^f_E(x_0)$. Since $x_n = \operatorname{proj}^f_{C_n}(x_0)$ and E is contained in C_n, we have $D_f\left(x_n, x_0\right) \leq D_f\left(\tilde{u}, x_0\right)$. Therefore Proposition 6 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u} = \operatorname{proj}^f_E(x_0)$, as claimed.

This completes the proof of Theorem 2. □

Now we present three consequences of Theorem 2. First we specialize to the case of one bifunction:

$$
\tag{4.3}
\begin{cases}
x_0 \in X, \\
C_0 = X, \\
y_n = \operatorname{Res}^f_{\lambda_n g}(x_n + e_n), \\
C_{n+1} = \{z \in C_n : D_f(z, y_n) \le D_f(z, x_n + e_n)\}, \\
x_{n+1} = \operatorname{proj}^f_{C_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases}
$$

In this case we obtain the following result as a direct consequence of Theorem 2.

Corollary 4. *Let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies conditions (C1)–(C4) with $EP(g) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (4.3). If $\liminf_{n\to+\infty} \lambda_n > 0$ and the sequence of errors $\{e_n\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to+\infty} e_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}^f_{EP(g)}(x_0)$ as $n \to +\infty$.*

The following corollary [**19**, Corollary 6] follows immediately from Theorem 2 when we take $\lambda^i_n = 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$.

Corollary 5. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (4.1). If, for each $i = 1, 2, \ldots, N$, the sequence of errors $\left\{e^i_n\right\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to+\infty} e^i_n = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}^f_E(x_0)$ as $n \to +\infty$.*

Theorem 2 holds, in particular, when the space X is both uniformly smooth and uniformly convex, and the function $f(x) = (1/2)\|x\|^2$. This leads us to the following result, which is a special case of Theorem 3.1 in [**24**]. More precisely, we consider following algorithm:

$$(4.4)\qquad \begin{cases} x_0 \in X, \\ C_0 = X, \\ y_n = \mathrm{Res}^f_{\lambda_n g}(x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

Corollary 6. *Let X be a uniformly smooth and uniformly convex Banach space, and let K be a nonempty, closed and convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a bifunction that satisfies conditions (C1)–(C4) with $EP(g) \neq \varnothing$. If $\liminf_{n\to+\infty} \lambda_n > 0$, then for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (4.4) converges strongly to $P_{EP(g)}(x_0)$ as $n \to +\infty$.*

5. Algorithm III

In this section we study a second algorithm based on the concept of the so-called shrinking projection method:

$$(5.1)\qquad \begin{cases} x_0 \in X, \\ Q_0^i = X, \qquad i = 1, 2, \ldots, N, \\ y_n^i = \mathrm{Res}^f_{\lambda_n^i g_i}(x_n + e_n^i), \\ Q_{n+1}^i = \left\{z \in Q_n^i : \left\langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \right\rangle \leq 0\right\}, \\ Q_{n+1} := \bigcap_{i=1}^N Q_{n+1}^i, \\ x_{n+1} = \mathrm{proj}^f_{Q_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots. \end{cases}$$

Theorem 3. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences*

$\{x_n\}_{n\in\mathbb{N}}$ which satisfy (5.1). If, for each $i=1,2,\ldots,N$, $\liminf_{n\to+\infty}\lambda_n^i>0$, and the sequence of errors $\left\{e_n^i\right\}_{n\in\mathbb{N}}\subset X$ satisfies $\lim_{n\to+\infty}e_n^i=0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_E^f(x_0)$ as $n\to+\infty$.

Proof. Our proof is again divided into four steps.

Step 1. *There are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (5.1).*

From Proposition 7(i) we know that $\operatorname{dom}\operatorname{Res}_{\lambda_n^i g_i}^f=X$ for any $i=1,2,\ldots,N$. Therefore each y_n^i is well-defined whenever x_n is. Let $n\in\mathbb{N}$. It is not difficult to check that the sets Q_n^i are closed and convex for all $i=1,2,\ldots,N$. Hence their intersection Q_n is also closed and convex. Let $u\in E$. For any $n\in\mathbb{N}$, we obtain from the definition of y_n^i that

$$\lambda_n^i g\left(y_n^i,u\right)+\left\langle\nabla f\left(y_n^i\right)-\nabla f\left(x_n+e_n^i\right),u-y_n^i\right\rangle\geq 0.$$

Since $u\in E$ and condition (C2) holds, we get

$$\left\langle\nabla f(x_n+e_n^i)-\nabla f(y_n^i),u-y_n^i\right\rangle\leq\lambda_n^i g\left(y_n^i,u\right)\leq-\lambda_n^i g\left(u,y_n^i\right)\leq 0,$$

which implies that $u\in Q_{n+1}^i$. Since this holds for any $i=1,2,\ldots,N$, it follows that $u\in Q_{n+1}$. Thus $E\subset Q_n$ for any $n\in\mathbb{N}$.

From now on we fix an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ which satisfies (5.1).

Step 2. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.*

It follows from Proposition 4(iii) that, for each $u\in E$, we have

$$\begin{aligned}D_f\left(x_n,x_0\right)&=D_f\left(\operatorname{proj}_{Q_n}^f(x_0),x_0\right)\\&\leq D_f\left(u,x_0\right)-D_f\left(u,\operatorname{proj}_{Q_n}^f(x_0)\right)\leq D_f\left(u,x_0\right).\end{aligned}$$

Hence the sequence $\{D_f(x_n,x_0)\}_{n\in\mathbb{N}}$ is bounded by $D_f(u,x_0)$ for any $u\in E$. Therefore by Proposition 5 the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too, as claimed.

Step 3. *Every weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to E.*

Since $x_{n+1}\in Q_{n+1}\subset Q_n$, it follows from Proposition 4(iii) that

$$D_f\left(x_{n+1},\operatorname{proj}_{Q_n}^f(x_0)\right)+D_f\left(\operatorname{proj}_{Q_n}^f(x_0),x_0\right)\leq D_f\left(x_{n+1},x_0\right)$$

and hence

$$D_f\left(x_{n+1},x_n\right)+D_f\left(x_n,x_0\right)\leq D_f\left(x_{n+1},x_0\right). \tag{5.2}$$

Therefore the sequence $\left\{D_f(x_n,x_0)\right\}_{n\in\mathbb{N}}$ is increasing and since it is also bounded (see Step 2), $\lim_{n\to+\infty}D_f(x_n,x_0)$ exists. Thus from (5.2) we obtain that

$$\lim_{n\to+\infty}D_f\left(x_{n+1},x_n\right)=0.$$

As in the proof of Theorem 1, it now follows that

$$\lim_{n\to+\infty}D_f\left(x_{n+1},x_n+e_n^i\right)=0. \tag{5.3}$$

For any $i=1,2,\ldots,N$, it follows from the inclusion $x_{n+1}\in Q_{n+1}^i$ that

$$\begin{aligned}0&\leq D_f\left(x_{n+1},y_n^i\right)+D_f\left(y_n^i,x_n+e_n^i\right)\\&\leq D_f\left(x_{n+1},y_n^i\right)+D_f\left(y_n^i,x_n+e_n^i\right)+\left\langle\nabla f(x_n+e_n^i)-\nabla f(y_n^i),y_n^i-x_{n+1}\right\rangle\\&=D_f\left(x_{n+1},x_n+e_n^i\right).\end{aligned}$$

From (5.3) we obtain that

$$\lim_{n\to+\infty}\left(D_f\left(x_{n+1},y_n^i\right)+D_f\left(y_n^i,x_n+e_n^i\right)\right)=0$$

and therefore $\lim_{n\to+\infty}D_f\left(x_{n+1},y_n^i\right)=0$.

Now, using an argument similar to the one we employed in the proof of Theorem 1 (see Step 3 there), we get the conclusion of Step 3.

Step 4. *The sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *converges strongly to* $\operatorname{proj}_E^f(x_0)$ *as* $n\to+\infty$.

From Proposition 7(v) it follows that $EP\left(g_i\right)$ is closed and convex for any $i=1,2,\ldots,N$. Therefore E is nonempty, closed and convex, and the Bregman projection proj_E^f is well defined. Let $\tilde{u}=\operatorname{proj}_E^f(x_0)$. Since $x_n=\operatorname{proj}_{Q_n}^f(x_0)$ and E is contained in Q_n, we know that $D_f\left(x_n,x_0\right)\leq D_f\left(\tilde{u},x_0\right)$. Therefore Proposition 6 implies that $\left\{x_n\right\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u}=\operatorname{proj}_E^f(x_0)$, as claimed.

This completes the proof of Theorem 3. □

Now we present three consequences of Theorem 3. In the first one (Corollary 7) there are no computational errors, in the second (Corollary 8) $\lambda_n^i = 1$ for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$, and in the third (Corollary 9) the space X is uniformly smooth and uniformly convex, and the function $f(x) = (1/2)\|x\|^2$. More precisely, we first consider the following algorithm:

$$
(5.4) \qquad \begin{cases}
x_0 \in X, \\
Q_0^i = X, \qquad i = 1, 2, \ldots, N, \\
y_n^i = \operatorname{Res}^f_{\lambda_n^i g_i}(x_n), \\
Q_{n+1}^i = \left\{ z \in Q_n^i : \left\langle \nabla f(x_n) - \nabla f(y_n^i), z - y_n^i \right\rangle \leq 0 \right\}, \\
Q_{n+1} := \bigcap_{i=1}^N Q_{n+1}^i, \\
x_{n+1} = \operatorname{proj}^f_{Q_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots.
\end{cases}
$$

In this case we obtain the following assertion as a direct consequence of Theorem 3.

Corollary 7. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (5.4). If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to+\infty} \lambda_n^i > 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}^f_E(x_0)$ as $n \to +\infty$.*

The next consequence of Theorem 3 is [**20**, Corollary 3].

Corollary 8. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) with $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy (5.1). If, for each $i = 1, 2, \ldots, N$, the sequence of* *errors* $\left\{e_n^i\right\}_{n\in\mathbb{N}} \subset X$ *satisfies* $\lim_{n\to+\infty} e_n^i = 0$, *then each such sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}_E^f(x_0)$ *as* $n \to +\infty$.

Finally, the third consequence of Theorem 3 concerns the following algorithm:

$$
\begin{cases}
x_0 \in X, \\
Q_0 = X, \\
y_n = \mathrm{Res}_{\lambda_n g}^f(x_n), \\
Q_{n+1} = \left\{z \in Q_n : \langle \nabla f(x_n) - \nabla f(y_n), z - y_n\rangle \le 0\right\}, \\
x_{n+1} = P_{Q_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots .
\end{cases} \tag{5.5}
$$

Corollary 9. *Let* X *be a uniformly smooth and uniformly convex Banach space, and let* K *be a nonempty, closed and convex subset of* X. *Let* $g : K \times K \to \mathbb{R}$ *be a bifunction that satisfies conditions (C1)–(C4) with* $EP(g) \neq \varnothing$. *If* $\liminf_{n\to+\infty} \lambda_n > 0$, *then for each* $x_0 \in X$, *the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *generated by (5.5) converges strongly to* $P_{EP(g)}(x_0)$ *as* $n \to +\infty$.

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References

[1] Ambrosetti, A. and Prodi, G.: A Primer of Nonlinear Analysis, *Cambridge University Press*, Cambridge, 1993.

[2] Bauschke, H. H. and Borwein, J. M.: Legendre functions and the method of random Bregman projections, *J. Convex Anal.* **4** (1997), 27–67.

[3] Bauschke, H. H., Borwein, J. M. and Combettes, P. L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Comm. Contemp. Math.* **3** (2001), 615–647.

[4] Bauschke, H. H., Borwein, J. M. and Combettes, P. L.: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* **42** (2003), 596–636.

[5] Bauschke, H. H. and Combettes, P. L.: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.* **26** (2001), 248–264.

[6] Bauschke, H. H., Wang, X. and Yao, L.: General resolvents for monotone operators: characterization and extension, *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems* (Huangguoshu 2008), to appear.

[7] Blum, E. and Oettli, W.: From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), 123–145.

[8] Bonnans, J. F. and Shapiro, A.: Perturbation Analysis of Optimization Problems, *Springer*, New York, 2000.

[9] Bregman, L. M.: The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming, *USSR Comput. Math. and Math. Phys.* **7** (1967), 200–217.

[10] Butnariu, D., Censor, Y. and Reich, S.: Iterative averaging of entropic projections for solving stochastic convex feasibility problems, *Comput. Optim. Appl.* **8** (1997), 21–39.

[11] Butnariu, D. and Iusem, A. N.: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.

[12] Butnariu, D., Iusem, A. N. and Resmerita, E.: Total convexity for powers of the norm in uniformly convex Banach spaces, *J. Convex Anal.* **7** (2000), 319–334.

[13] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.* **2006** (2006), Art. ID 84919, 1–39.

[14] Censor, Y. and Lent, A.: An iterative row-action method for interval convex programming, *J. Optim. Theory Appl.* **34** (1981), 321–353.

[15] Combettes, P. L. and Hirstoaga, S. A.: Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* **6** (2005), 117–136.

[16] Reich, S. and Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces, *J. Nonlinear Convex Anal.* **10** (2009), 471–485.

[17] Reich, S. and Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces, *Numer. Funct. Anal. Optim.* **31** (2010), 22–44.

[18] Reich, S. and Sabach, S.: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces, *Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer*, New York, accepted for publication.

[19] Reich, S. and Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, *Nonlinear Analysis* **73** (2010), 122–135.

[20] Reich, S. and Sabach, S.: A shrinking projection method for Bregman firmly nonexpansive operators in reflexive Banach spaces, preprint, 2010.

[21] Resmerita, E.: On total convexity, Bregman projections and stability in Banach spaces, *J. Convex Anal.* **11** (2004), 1–16.

[22] Solodov, M. V. and Svaiter, B. F.: Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Program.* **87** (2000), 189–202.

[23] Takahashi, W., Takeuchi, Y. and Kubota, R.: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **341** (2008), 276–286.

[24] Takahashi, W. and Zembayashi, K.: Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, *Fixed point Theory Appl.* **2008** (2008), Art. ID 528476, 1-11.

[25] Takahashi, W. and Zembayashi, K.: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **70** (2009), 45–57.

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