Products of Finitely Many Resolvents of Maximal Monotone Mappings in Reflexive Banach Spaces

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Abstract. We propose two algorithms for finding (common) zeroes of finitely many maximal monotone mappings in reflexive Banach spaces. These algorithms are based on the Bregman distance related to a well-chosen convex function and improves previous results.

Finally, we mention two applications of our algorithms for solving equilibrium problems and convex feasibility problems.

1. Introduction

In this paper we are concerned with the problem of finding zeroes of mappings $A : X \to 2^{X^*}$, that is, find $x \in \operatorname{dom} A$ such that

$$0^* \in Ax. \tag{1.1}$$

The *domain* of a mapping A is defined by the set $\{x \in X : Ax \neq \varnothing\}$.

Many problems have reformulations which require to find zeroes, for instance, differential equations, evolution equations, complementarity problems, mini-max problems, variational inequalities and optimization problems. It is well known that minimizing a convex function f can be reduced to finding zeroes of the subdifferential mapping $A = \partial f$.

One of the most important techniques for solving the inclusion (1.1) is going back to the work of Browder [**15**] in the sixties. One of the basic ideas in the case of a Hilbert space H is reducing (1.1) to a fixed point problem of the operator $R_A : H \to 2^H$ defined by

$$R_A = (I + A)^{-1}$$

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which we call in what follows the *classical resolvent* of A. When H is a Hilbert space and A satisfies some monotonicity conditions (see Section 2.2), the classical resolvent of A is with full domain and *nonexpansive*, that is,

$$\|R_A x - R_A y\| \leq \|x - y\| \quad \forall\, x, y \in H,$$

and even *firmly nonexpansive*, that is,

$$\|R_A x - R_A y\|^2 \leq \langle R_A x - R_A y, x - y\rangle \quad \forall\, x, y \in H.$$

These properties of the resolvent ensure that its Picard iterates $x_{n+1} = R_A x_n$ converge weakly, and sometimes even strongly, to a fixed point of R_A which is necessarily a zero of A. Rockafellar introduced this iteration method and called it the *proximal point algorithm* (see [**44, 45**]).

Methods for finding zeroes of monotone mappings in Hilbert space are based on the good properties of the resolvent R_A such as nonexpansiveness but when we try to extend these methods to Banach spaces we encounter several difficulties (see, for example, [**21**]).

One way to overcome this difficulty is to use, instead of the classical resolvent, a new type of resolvent introduced by Bauschke, Borwein and Combettes (see [**5**]). If $f : X \to (-\infty, +\infty]$ is a Legendre (see Section 2.1) and convex function, then the operator $\mathrm{Res}_A^f : X \to 2^X$ given by

$$\mathrm{Res}_A^f = (\nabla f + A)^{-1} \circ \nabla f$$

is well defined when A is maximal monotone and $\operatorname{int}\operatorname{dom} f \bigcap \operatorname{dom} A \neq \varnothing$. Moreover, similarly to the classical resolvent, a fixed point of Res_A^f is a solution of (1.1). This leads to the question whether, and under which conditions for A and f, the iterates of Res_A^f approximate a fixed point of Res_A^f.

In order to modify the proximal point algorithm for the new resolvent and prove the convergence of the iterates of Res_A^f we need nonexpansivity properties of this resolvent as in the case of the classical resolvent. Bauschke, Borwein and Combettes introduced the class of Bregman firmly nonexpansive operators (see Subsection 2.3) and proved that the resolvent, Res_A^f, belongs to this class. They also proved many others properties of this resolvent (see [**5**]). These properties are essential to the convergence of the iterates of Res_A^f. There are many papers that deal with the proximal point algorithm in Hilbert and Banach spaces (see, for instance, [**6, 9, 10, 14, 20, 36, 37**]).

In this paper we propose two algorithms using products of resolvents. We modify the classical proximal point algorithm in order to obtain the strong convergence of the iterates of Res_A^f. These algorithms hold in general reflexive Banach spaces and take into account possible computational errors.

Our paper is organized as follows. In the following section we give a brief overview of the concepts we will use further. Section 2 consists of three subsections where the first one deals with functions and the second deals with mappings. The third subsection focused on types of Bregman nonexpansive operators. We introduce our two algorithms and prove the main result in the third section (Theorem 3.1). Sections 4 and 5 include applications of Theorem 3.1. We modify our algorithms in order to solve equilibrium problems (Theorem 4.1) and propose algorithms for solving the convex feasibility problem (Theorem 5.1).

2. Preliminaries

Throughout this paper X is a real reflexive Banach space and X^* is its dual. For $\xi \in X^*$ and $x \in X$ the pairing $\langle \xi, x \rangle$ denotes the value of ξ at x. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of nonnegative integers.

We divide our preliminaries into three subsections. The first one (Subsection 2.1) is devoted to notions and results on functions. In the second subsection (Subsection 2.2) we give notions and basic results of mappings. The last subsection (Subsection 2.3) deals with types of Bregman nonexpansive operators.

2.1. Properties of Functions. Let $f : X \to (-\infty, +\infty]$ be a function. The *domain* of the function f is the set

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

If $\operatorname{dom} f \neq \varnothing$ then the function f is called *proper*. From now on, we assume that f is a proper, convex and lower semicontinuous function.

Let $x \in \operatorname{int} \operatorname{dom} f$. The function f is called *Gâteaux differentiable* at x if

$$f^{\circ}(x, y) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \tag{2.1}$$

exists for any $y \in X$. In this case, the *gradient* of f at x is defined by $\nabla f(x) := f^{\circ}(x, \cdot)$. If f is Gâteaux differentiable at any $x \in \operatorname{int} \operatorname{dom} f$ we will say that f is Gâteaux differentiable.

If the limit in (2.1) is attained uniformly for any $y \in X$ with $\|y\| = 1$ we say that f is *Fréchet differentiable* at x. Let E be a subset of X. If the limit in (2.1) is attained uniformly for any $x \in E$ and $y \in X$ with $\|y\| = 1$ we say that f is *uniformly Fréchet differentiable* at x.

It is well known that f is Gâteaux (respectively Fréchet) differentiable at $x \in \operatorname{int} \operatorname{dom} f$ if and only if the gradient ∇f is norm-to-weak* (norm-to-norm) continuous at x (see [**34**, Propostion 2.8, p. 19]). In [**36**] the authors proved the following result which will be very useful in the proof of our main result (Theorem 3.1).

PROPOSITION 2.1 (*cf.* [**36**, Proposition 2.1, p. 474]). *If $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

The function $f : X \to (-\infty, +\infty]$ is called *Legendre* if it satisfies the following two conditions:
(L1) the function f is Gâteaux differentiable, $\operatorname{int}\operatorname{dom} f \neq \varnothing$ and $\operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$;
(L2) the function f^* is Gâteaux differentiable, $\operatorname{int}\operatorname{dom} f^* \neq \varnothing$ and $\operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

The notion of Legendre functions in infinite dimensional spaces was first studied by Bauschke, Borwein and Combettes in [**4**]. It is clear from the definition that f is Legendre if and only if f^* is Legendre. It also follows that f and f^* are strictly convex on $\operatorname{int}\operatorname{dom} f$ and $\operatorname{int}\operatorname{dom} f^*$, respectively. The function $(1/p) \|\cdot\|^p$ is Legendre for any $1 < p < +\infty$ when the Banach space is smooth and strictly convex. Other properties and examples of Legendre functions can be found, for example, in [**3, 4**].

From now on we assume that $f : X \to (-\infty, +\infty]$ is also Legendre.

The Bregman distance with respect to f, or simply, *Bregman distance* which was introduced in [**22**] is the bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle . \tag{2.2}$$

Its importance in optimization as a substitute for the usual distance or, more exactly, for the square of the norm-induced on X, was first emphasized by Bregman [**13**]. It should be noted that D_f is not a distance in the usual sense of the term. Clearly, $D_f(x, x) = 0$, but $D_f(y, x) = 0$ may not imply $x = y$. In our case when f is Legendre this indeed holds (see [**4**, Theorem 7.3(vi), p. 642]). In general, D_f is not symmetric and does not satisfies the triangle inequality. However, D_f satisfies the *three point identity*

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle . \tag{2.3}$$

for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$.

The *modulus of total convexity* at x is the bifunction $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$, defined by

$$\upsilon_f(x, t) := \inf \{ D_f(y, x) : y \in \operatorname{dom}\ f,\ \|y - x\| = t \} .$$

The function f is called *totally convex* at $x \in \operatorname{int}\operatorname{dom} f$ if $\upsilon_f(x, t)$ is positive for any $t > 0$. This notion was first introduced by Butnariu and Iusem in [**18**, Section

1.2, p. 17] (see also [**17**]). Let E be a nonempty subset of X. The *modulus of total convexity* of f on E is the bifunction $\upsilon_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\upsilon_f(E, t) := \inf\left\{\upsilon_f(x, t) : x \in E \bigcap \operatorname{int}\operatorname{dom} f\right\}.$$

The function f is called *totally convex on bounded subsets* if $\upsilon_f(E, t)$ is positive for any nonempty and bounded subset E and for any $t > 0$.

In [**18**, Proposition 1.2.5, p. 25] the authors proved that any uniformly convex function at $x \in \operatorname{int}\operatorname{dom} f$ (see [**49**]) is totally convex function at $x \in \operatorname{int}\operatorname{dom} f$. It is also known that every totally convex function is strictly convex. For the case of bounded subsets there is no difference between these notions, that is, f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [**21**, Theorem 2.10, p. 9]). For more information on totally convex functions and Bregman distance see, for instance, [**18, 19, 43**].

We have the following property of the Bregman distance related to totally convex functions.

PROPOSITION 2.2 (*cf.* [**37**, Lemma 3.1, p. 31]). *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x \in X$ and the sequence $\{D_f(x_n, x)\}_{n\in\mathbb{N}}$ is bounded, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too.*

The next property is also true where ∇f^* is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} f^*$.

PROPOSITION 2.3. *Let $f : X \to \mathbb{R}$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *Let $x \in X$ if $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$ is bounded, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded too.*

PROOF. Combining Theorem 3.3 and Lemma 7.3(viii) of [**4**] with Proposition 4.1(v)(a) of [**5**]. □

Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, where the first one is bounded. The function f is called *sequentially consistent* if

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The following result emphasizes the connection between the notions of sequentially consistent and totally convex functions on bounded subsets.

PROPOSITION 2.4 (*cf.* [**18**, Lemma 2.1.2, p. 67]). *The function f is totally convex on bounded subsets if and only if it is sequentially consistent.*

2.2. Properties of Mappings. Let $A : X \to 2^{X^*}$ be a mapping. The *resolvent* of A is the multi-valued operator $\operatorname{Res}_A^f : X \to 2^X$ defined by

$$\operatorname{Res}_A^f = (\nabla f + A)^{-1} \circ \nabla f. \tag{2.4}$$

This class of operators was first introduced and studied in [**5**]. If we assume that the mapping A is *monotone*, that is, satisfies the following inequality

$$\langle \xi - \eta, x - y \rangle \geq 0. \tag{2.5}$$

for any $\xi \in Ax$ and $\eta \in Ay$, then the resolvent is single-valued when f is strictly convex on $\operatorname{int}\operatorname{dom} f$ (as in our case). If $A = \partial\varphi$, where φ is a proper, lower semicontinuous and convex function, then the proximal operator $\operatorname{prox}_{\varphi}^{f}$ is defined as follows

$$\operatorname{prox}_{\varphi}^{f} := \operatorname{Res}_{\partial\varphi}^{f}.$$

If K is a subset of X then the *indicator function* ι_K of K is defined by

$$\iota_K(x) := \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K. \end{cases}$$

When the set K is nonempty, closed and convex the function ι_K is proper, convex and lower semicontinuous, and therefore $\partial\iota_K$ exists and is a maximal monotone mapping with domain K (see [**24**, Proposition 4.1, p. 168]). We recall that, a monotone mapping A is said to be *maximal* if graph A is not a proper subset of the graph of any other monotone mapping.

If we take $\varphi = \iota_K$ then the proximal operator $\operatorname{prox}_{\iota_K}^{f}$ is called the *Bregman projection* onto K with respect to f and we denote it by proj_K^f. Therefore, the Bregman projection (*cf.* [**13**]) of $x \in \operatorname{int}\operatorname{dom} f$ onto a nonempty, closed and convex subset K of $\operatorname{dom} f$ is necessarily the unique vector $\operatorname{proj}_K^f(x) \in K$ which satisfies

$$D_f\left(\operatorname{proj}_K^f(x), x\right) = \inf\left\{D_f(y, x) : y \in K\right\}.$$

This projection is a generalization for Banach spaces of the metric projection in Hilbert spaces. Indeed, when X is a Hilbert space and $f = \|\cdot\|^2$, then the Bregman distance $D_f(y,x)$ equals $\|y - x\|^2$ and the Bregman projection of x onto K is the *metric projection* P_K, *i.e.*, $\operatorname{argmin}\{\|y - x\| : y \in K\}$.

In addition to this similarity between Bregman and metric projection we have the following variational characterization of Bregman projection which is a generalization of the metric projection characterization.

Proposition 2.5 (*cf.* [**21**, Corollary 4.4, p. 23]). *Suppose that f is Gâteaux differentiable and totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $K \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed and convex set. If $\hat{x} \in K$, then the following conditions are equivalent: (i) the vector $\hat{x}$ is the Bregman projection of x onto K with respect to f;*

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \qquad \forall y \in K;$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,z) + D_f(z,x) \le D_f(y,x) \qquad \forall y \in K.$$

The next result is essential for our purposes.

Proposition 2.6 (*cf.* [**37**, Lemma 3.2, p. 31]). *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in X$ and let K be a nonempty, closed and convex subset of X. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and any weak subsequential limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to K. If $D_f(x_n, x_0) \le D_f\left(\operatorname{proj}_K^f(x_0), x_0\right)$ for any $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_K^f(x_0)$.*

2.3. Types of Bregman Nonexpansive Operators. In 2003, Bauschke, Borwein and Combettes [**5**] studied the following class of operators. Let K be a nonempty subset of $\operatorname{int}\operatorname{dom} f$ and let T be an operator from K to $\operatorname{int}\operatorname{dom} f$. An operator T is called *Bregman firmly nonexpansive* (BFNE) if it satisfies the following inequality

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \tag{2.6}$$

for all $x, y \in K$. Inequality (2.6) is equivalent to the following inequality

$$D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \le D_f(Tx,y) + D_f(Ty,x) \tag{2.7}$$

since we use the definition of Bregman distance (see (2.2)). More information on BFNE operators can be found, for example, in [**5, 38**].

The *fixed point set* of the operator T is denoted by $F(T)$. If $F(T)$ is nonempty then we can take $x \in K$ and $p := y \in F(T)$ in (2.6) and obtain the following inequality

$$\langle \nabla f(x) - \nabla f(Tx), Tx - p\rangle \ge 0 \tag{2.8}$$

which is equivalent to

$$D_f(p,Tx) + D_f(Tx,x) \le D_f(p,x). \tag{2.9}$$

An operator which satisfies (2.8) is called *quasi-Bregman firmly nonexpansive* (QBFNE). Any QBFNE operator satisfies

$$D_f(p,Tx) \le D_f(p,x) \tag{2.10}$$

for all $x \in K$ and $p \in F(T)$. Any operator which satisfies (2.10) will be called *quasi-Bregman nonexpansive* (QBNE). In [**38**, Lemma 15.5, p. 305] it is proved that the fixed point set of a QBNE operator is closed and convex when f is a Legendre function. Bauschke, Borwein and Combettes [**5**] proved that when the mapping A is maximal monotone then its resolvent Res_A^f is a BFNE single valued operator with full domain and

$$F\left(\operatorname{Res}_A^f\right) = A^{-1}\left(0^*\right) \bigcap \left(\operatorname{int}\operatorname{dom} f\right)$$

where $x \in A^{-1}\left(0^*\right)$ if and only if $0^* \in Ax$.

Let $f : X \to \mathbb{R}$ be a Legendre function and assume that $A^{-1}\left(0^*\right) \neq \varnothing$. Then $F\left(\operatorname{Res}_A^f\right) = A^{-1}\left(0^*\right) \neq \varnothing$ and the resolvent is also a QBNE operator (since any BFNE operator is QBNE when its fixed point set is nonempty) and therefore

$$D_f\left(u, \operatorname{Res}_A^f(x)\right) \leq D_f(u, x) \tag{2.11}$$

for all $u \in A^{-1}\left(0^*\right)$ and $x \in X$.

Let K be a nonempty subset of X. A point $p \in K$ is said to be an *asymptotic fixed point* of T [**35**] if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in K such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. We denote the *asymptotic fixed point set* of T by $\hat{F}(T)$. It is clear from the definition that $F(T) \subset \hat{F}(T)$ for any operator T.

Another class of operators which was introduced in [**23, 35**] is the following. We say that an operator T is *Bregman strongly nonexpansive* (BSNE) with respect to a nonempty $\hat{F}(T)$ if for all $p \in \hat{F}(T)$ and $x \in K$

$$D_f(p, Tx) \leq D_f(p, x) \tag{2.12}$$

and for any $\{x_n\}_{n \in \mathbb{N}} \subset K$ bounded, $p \in \hat{F}(T)$, with

$$\lim_{n \to \infty}\left(D_f(p, x_n) - D_f(p, Tx_n)\right) = 0, \tag{2.13}$$

it follows that

$$\lim_{n \to \infty} D_f(Tx_n, x_n) = 0. \tag{2.14}$$

Note that the notion of strongly nonexpansive operators (with respect to the norm) was first introduced and studied in [**16**]. Reich proves in [**35**] two properties of BSNE operators. These two properties are summarized in the following result.

Proposition 2.7. *Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let K be a nonempty, closed and convex subset of X. If $\{T_i : 1 \leq i \leq N\}$ are N BSNE operators from K into itself, and the set*

$$\hat{F} = \bigcap \left\{\hat{F}(T_i) : 1 \leq i \leq N\right\}$$

is not empty, then $\hat{F}(T_N T_{N-1} \cdots T_1) \subset \hat{F}$. In addition, the operator $T_N T_{N-1} \cdots T_1$ is also BSNE operator with respect to $\hat{F}(T_N T_{N-1} \cdots T_1)$.

In applications it seems that the assumption $\hat{F}(T) = F(T)$ on the operator T is essential for the convergence of iterative methods. In [**38**, Lemma 15.6, p. 306] a sufficient condition for a BFNE operator to satisfy this condition is given.

Remark 1. *Let* $\{T_i : 1 \le i \le N\}$ *be* N *BSNE operators which satisfy* $\hat{F}(T_i) = F(T_i)$ *for each* $1 \le i \le N$ *and let* $T = T_N T_{N-1} \cdots T_1$*. If*

$$F = \bigcap \{F(T_i) : 1 \le i \le N\}$$

and $F(T)$ *are nonempty, then* T *is also BSNE with* $F(T) = \hat{F}(T)$*. Indeed, from Proposition 2.7 we get that*

$$F(T) \subset \hat{F}(T) \subset \bigcap \left\{\hat{F}(T_i) : 1 \le i \le N\right\} = \bigcap \{F(T_i) : 1 \le i \le N\} \subset F(T),$$

which implies that $F(T) = \hat{F}(T)$*, as claimed.*

3. Products of Resolvents

In this section we propose two algorithms for finding common zeroes of finitely many maximal monotone mappings. Both algorithms are based on products of resolvents. For earlier results based on this method see, for example, [**9, 16, 35, 42**].

3.1. Bauschke-Combettes Iterative Method.

As we have seen in the Introduction, one of the main methods for finding zeroes of maximal monotone mapping in Hilbert spaces, is the classical proximal point algorithm

$$x_{n+1} = R_{\lambda_n A}(x_n) = (I + \lambda_n A)^{-1} x_n, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

where $\{\lambda_n\}_{n \in \mathbb{N}}$ is a given sequence of positive real numbers. Note that (3.1) is equivalent to

$$0 \in Ax_{n+1} + \frac{1}{\lambda_n}(x_{n+1} - x_n), \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

This algorithm was first introduced by Martinet [**33**] and further developed by Rockafellar [**44**], who proved that the sequence generated by (3.1) converges weakly to an element of $A^{-1}(0)$ when $A^{-1}(0)$ is nonempty and $\liminf_{n\to\infty} \lambda_n > 0$. Furthermore, Rockafellar [**44**] raised the question whether sequence generated by (3.1) converges strongly. For general monotone mappings a negative answer to this question follows from [**28**]; see also [**10**]. In the case of subdifferentials this question was answered in the negative by Güler [**29**], who presented an example of a subdifferential mapping for which the sequence generated by (3.1) converges weakly but not strongly (see [**10**] for a more recent and simpler example). Bauschke and Combettes [**6**] have modified the proximal point algorithm (see (3.1) and (3.2)) in order to generate a strongly convergent sequence. They introduced, for example, the following algorithm (see [**6**, Corollary 6.1(ii), p. 258] for a single mapping and

$\lambda_n = 1/2$):

$$
\left\{
\begin{array}{l}
x_0 \in H, \\
y_n = R_{\lambda_n A}\left(x_n\right), \\
C_n = \left\{z \in H : \left\|y_n - z\right\| \leq \left\|x_n - z\right\|\right\}, \\
Q_n = \left\{z \in H : \left\langle x_0 - x_n, z - x_n\right\rangle \leq 0\right\}, \\
x_{n+1} = P_{C_n \cap Q_n}\left(x_0\right), \qquad n = 0, 1, 2, \ldots.
\end{array}
\right.
\tag{3.3}
$$

Recently, this algorithm was generalized to general reflexive Banach spaces in the following way (see [**37**]).

$$
\left\{
\begin{array}{l}
x_0 \in X, \\
y_n = \mathrm{Res}^f_{\lambda_n A}\left(x_n\right), \\
C_n = \left\{z \in X : D_f\left(z, y_n\right) \leq D_f\left(z, x_n\right)\right\}, \\
Q_n = \left\{z \in X : \left\langle \nabla f\left(x_0\right) - \nabla f\left(x_n\right), z - x_n\right\rangle \leq 0\right\}, \\
x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}\left(x_0\right), \qquad n = 0, 1, 2, \ldots.
\end{array}
\right.
\tag{3.4}
$$

where $f : X \to (-\infty, +\infty]$ is a well chosen convex function.

In [**37**] we proposed a modification of Algorithm (3.4) for finding common zeroes of finitely many maximal monotone mappings. In this algorithm we build, at any step, N copies of the half-space C_n with respect to any mapping. Then the next iteration is the Bregman projection onto the intersection of $N+1$ half-spaces (N copies of C_n and Q_n). In this paper we propose a new variant of Algorithm (3.4) which also find common zeroes of finitely many maximal monotone mappings. In the new algorithm we use the concept of products of resolvents and therefore we build, at any step, only one copy of the half-space C_n. Then the the next iteration is the Bregman projection onto the intersection of two half-spaces (C_n and Q_n).

$$
\left\{
\begin{array}{l}
x_0 \in X, \\
y_n = \mathrm{Res}^f_{\lambda_n^N A_N} \cdots \mathrm{Res}^f_{\lambda_n^1 A_1}\left(x_n + e_n\right), \\
C_n = \left\{z \in X : D_f\left(z, y_n\right) \leq D_f\left(z, x_n + e_n\right)\right\}, \\
Q_n = \left\{z \in X : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n\right\rangle \leq 0\right\}, \\
x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0), \qquad n = 0, 1, 2, \ldots,
\end{array}
\right.
\tag{3.5}
$$

The following algorithm is a modification of Algorithm (3.5) where at any step we calculate the Bregman projection onto only one set which is not a half-space. Even if we only project onto one set, the computation of the projection is harder since this set is a general convex set. Even though we present and analyze this algorithm since the proof is very similar to the one of Algorithm (3.5). More precisely, we introduce the following algorithm.

$$
\left\{
\begin{array}{l}
x_0 \in X, \\
H_0 = X, \\
y_n = \mathrm{Res}^f_{\lambda_n^N A_N} \cdots \mathrm{Res}^f_{\lambda_n^1 A_1} \left(x_n + e_n\right), \\
H_{n+1} = \left\{z \in H_n : D_f\left(z, y_n\right) \leq D_f\left(z, x_n + e_n\right)\right\}, \\
x_{n+1} = \mathrm{proj}^f_{H_{n+1}}\left(x_0\right), \qquad n = 0, 1, 2, \ldots.
\end{array}
\right. \tag{3.6}
$$

We have the following theorem.

Theorem 3.1. *Let* $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$, *be* N *maximal monotone mappings with* $Z := \bigcap_{i=1}^{N} A_i^{-1}\left(0^*\right) \neq \varnothing$. *Let* $f : X \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X. *Suppose that* ∇f^* *is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *Then, for each* $x_0 \in X$, *the sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *which is generated by (3.5) or (3.6) is well defined. If the sequence of errors* $\left\{e_n\right\}_{n\in\mathbb{N}} \subset X$ *satisfies* $\lim_{n\to\infty} e_n = 0$ *and for each* $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, *then the sequence* $\left\{x_n\right\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}^f_Z\left(x_0\right)$ *as* $n \to \infty$.

We will prove Theorem 3.1 by sequence of five lemmata.

Lemma 3.1. *Algorithms (3.5) and (3.6) are well defined.*

Proof. We have to prove that the sequences $\left\{x_n\right\}_{n\in\mathbb{N}}$ and $\left\{y_n\right\}_{n\in\mathbb{N}}$ are well defined, that is, we have to prove that the Bregman projection onto $C_n \bigcap Q_n$, H_n and Z are well defined. We will show that these sets are nonempty, closed and convex.

As we explained in Section 2.2 we have that $F\left(\mathrm{Res}^f_{\lambda A_i}\right) = A_i^{-1}\left(0^*\right)$ is closed and convex for any $i = 1, 2, \ldots, N$. Therefore Z is nonempty, closed and convex and the Bregman projection onto Z, proj^f_Z, is well defined.

Note that $\operatorname{dom}\nabla f = X$ because $\operatorname{dom} f = X$ and f is Legendre. Hence it follows from [**5**, Corollary 3.14, p. 606] that $\operatorname{dom}\mathrm{Res}^f_{\lambda A} = X$. Hence the sequence $\left\{y_n\right\}_{n\in\mathbb{N}}$ is well defined. It is easy to check that C_n is a closed half-space for any $n \in \mathbb{N}$. Let $u \in Z$. We denote by T_n^i the resolvent $\mathrm{Res}^f_{\lambda_n^i A_i}$ and by S_n^i the composition $T_n^i \cdots T_n^1$ for any $i = 1, 2, \ldots, N$ and for each $n \in \mathbb{N}$. Therefore $y_n = T_n^N \cdots T_n^1\left(x_n + e_n\right) = S_n^N\left(x_n + e_n\right)$. We also assume that $S_n^0 = I$, where I is the identity operator.

Each resolvent $\mathrm{Res}^f_{\lambda_n^i A_i}$ is a QBNE operator and therefore S_n^N, a composition of QBNE operators, is also QBNE. Hence we get from (2.12) that

$$
\begin{aligned}
D_f\left(u, y_n\right) &= D_f\left(u, \mathrm{Res}^f_{\lambda_n^N A_N} \cdots \mathrm{Res}^f_{\lambda_n^1 A_1}\left(x_n + e_n\right)\right) = D_f\left(u, S_n^N\left(x_n + e_n\right)\right) \\
&\leq D_f\left(u, x_n + e_n\right) \qquad (3.7)
\end{aligned}
$$

which implies that $u \in C_n$. Thus $Z \subset C_n$ for any $n \in \mathbb{N}$. In the same way we prove that $Z \subset H_n$ for any $n \in \mathbb{N}$. This proves that Algorithm (3.6) is well defined.

For Algorithm (3.5) we only have to show that $C_n \bigcap Q_n$ is nonempty. We will prove that by induction. It is clear that $Z \subset Q_0 = X$. Thus $Z \subset C_0 \bigcap Q_0$. Now suppose that $Z \subset C_{n-1} \bigcap Q_{n-1}$ for some $n \geq 1$. Then $x_n = \operatorname{proj}^f_{C_{n-1} \cap Q_{n-1}}(x_0)$ is well defined because $C_{n-1} \bigcap Q_{n-1}$ is a nonempty, closed and convex subset of X. Thence from Proposition 2.5(ii) we obtain

$$\langle \nabla f(x_0) - \nabla f(x_n), y - x_n \rangle \leq 0$$

for any $y \in C_{n-1} \bigcap Q_{n-1}$. Hence $Z \subset Q_n$ and therefore $Z \subset C_n \bigcap Q_n$. Consequently, we get that $Z \subset C_n \bigcap Q_n$ for any $n \in \mathbb{N}$. This proves that Algorithm (3.5) is well defined, as claimed. □

Lemma 3.2. *Let $\{x_n\}_{n \in \mathbb{N}}$ be the sequence which is generated by Algorithm (3.5) or (3.6). Then the sequences $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ and $\{x_n\}_{n \in \mathbb{N}}$ are bounded.*

Proof. We start with Algorithm (3.5). It is easy to see that $x_n = \operatorname{proj}^f_{Q_n}(x_0)$ which means that we can use Proposition 2.5(iii) with $K = Q_n$. Therefore

$$D_f(x_n, x_0) = D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f(u, x_0) - D_f\left(u, \operatorname{proj}^f_{Q_n}(x_0)\right) \leq D_f(u, x_0) \tag{3.8}$$

for all $u \in Z \subset Q_n$. In Algorithm (3.6) the situation is similar where $K = H_n$ and $x_n = \operatorname{proj}^f_{H_n}(x_0)$. Therefore (3.8)

$$D_f(x_n, x_0) = D_f\left(\operatorname{proj}^f_{H_n}(x_0), x_0\right) \leq D_f(u, x_0) - D_f\left(u, \operatorname{proj}^f_{H_n}(x_0)\right) \leq D_f(u, x_0) \tag{3.9}$$

for any $u \in Z \subset H_n$.

Hence in both cases the sequence $\{D_f(x_n, x_0)\}_{n \in \mathbb{N}}$ is bounded, as asserted. Now we use Proposition 2.2 in order to get the second desired result. □

Lemma 3.3. *Let $\{x_n\}_{n \in \mathbb{N}}$ be the sequence which is generated by Algorithm (3.5) or (3.6). Then the sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\left\{S^i_n(x_n + e_n)\right\}_{n \in \mathbb{N}}$, $i = 1, 2, \ldots, N$, are bounded.*

Proof. Let $u \in Z$. From the three point identity (see 2.3)) we get that

$$\begin{aligned} D_f(u, x_n + e_n) &= D_f(u, x_n) - D_f(x_n + e_n, x_n) \\ &\quad + \langle \nabla f(x_n + e_n) - \nabla f(x_n), u - (x_n + e_n) \rangle \\ &\leq D_f(u, x_n) + \langle \nabla f(x_n + e_n) - \nabla f(x_n), u - (x_n + e_n) \rangle . \end{aligned} \tag{3.10}$$

In the case of Algorithm (3.5) we have that

$$D_f(u, x_n) = D_f\left(u, \operatorname{proj}^f_{C_{n-1} \cap Q_{n-1}}(x_0)\right) \leq D_f(u, x_0)$$

since the Bregman projection is QBNE and $Z \subset C_{n-1} \bigcap Q_{n-1}$. In the case of Algorithm (3.6) the situation is similar since $Z \subset H_n$. Therefore in both cases we have that

$$D_f\left(u, x_n\right) \leq D_f\left(u, x_0\right). \tag{3.11}$$

On the other hand since the sequence $\left\{x_n\right\}_{n \in \mathbb{N}}$ is bounded (see Lemma 3.2) and f is uniformly Fréchet differentiable we obtain from Proposition 2.1 that

$$\lim_{n \to \infty} \left\| \nabla f\left(x_n + e_n\right) - \nabla f\left(x_n\right) \right\|_* = 0$$

because $\lim_{n \to \infty} e_n = 0$. This means that if we take into account that $\left\{x_n\right\}_{n \in \mathbb{N}}$ is bounded we get

$$\lim_{n \to \infty} \left\langle \nabla f\left(x_n\right) - \nabla f\left(x_n + e_n\right), u - \left(x_n + e_n\right) \right\rangle = 0. \tag{3.12}$$

Combining (3.10), (3.11) and (3.12) shows that $\left\{D_f\left(u, x_n + e_n\right)\right\}_{n \in \mathbb{N}}$ is bounded. Now from (3.7) we see that also $\left\{D_f\left(u, y_n\right)\right\}_{n \in \mathbb{N}}$ is bounded. The boundedness of the sequence $\left\{y_n\right\}_{n \in \mathbb{N}}$ now follows from Proposition 2.3. In addition, we have for any $i = 1, 2, \ldots, N$ that

$$D_f\left(u, S_n^i\left(x_n + e_n\right)\right) \leq D_f\left(u, x_n + e_n\right). \tag{3.13}$$

Therefore in a similar way we prove that each $\left\{S_n^i\left(x_n + e_n\right)\right\}_{n \in \mathbb{N}}$ is bounded. $\square$

Lemma 3.4. *Every weak subsequential limit of $\left\{x_n\right\}_{n \in \mathbb{N}}$ which is generated by Algorithm (3.5) or (3.6) belongs to Z.*

Proof. We will show that both algorithms satisfy

$$D_f\left(x_{n+1}, x_n\right) + D_f\left(x_n, x_0\right) \leq D_f\left(x_{n+1}, x_0\right). \tag{3.14}$$

In Algorithm (3.5) it follows from the definition of Q_n and Proposition 2.5(ii) that $\mathrm{proj}_{Q_n}^f\left(x_0\right) = x_n$. Since $x_{n+1} \in Q_n$, it follows from Proposition 2.5(iii) that

$$D_f\left(x_{n+1}, \mathrm{proj}_{Q_n}^f\left(x_0\right)\right) + D_f\left(\mathrm{proj}_{Q_n}^f\left(x_0\right), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and therefore (3.14) holds. In Algorithm (3.6) it follows from the fact that $x_{n+1} = \mathrm{proj}_{H_{n+1}}^f\left(x_0\right) \in H_{n+1} \subset H_n$ and again from Proposition 2.5(iii) we get that

$$D_f\left(x_{n+1}, \mathrm{proj}_{H_n}^f\left(x_0\right)\right) + D_f\left(\mathrm{proj}_{H_n}^f\left(x_0\right), x_0\right) \leq D_f\left(x_{n+1}, x_0\right)$$

and therefore (3.14) holds.

Therefore the sequence $\left\{D_f\left(x_n, x_0\right)\right\}_{n \in \mathbb{N}}$ is increasing and since it is also bounded (see Lemma 3.2), $\lim_{n \to \infty} D_f\left(x_n, x_0\right)$ exists. Thus from (3.14) it follows that

$$\lim_{n \to \infty} D_f\left(x_{n+1}, x_n\right) = 0. \tag{3.15}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.2), Proposition 2.4 now implies that $\lim_{n\to\infty}(x_{n+1}-x_n)=0$.

It follows from the definition of the Bregman distance (see (2.2)) that

$$\begin{aligned} D_f\left(x_n, x_n+e_n\right) &= f\left(x_n\right)-f\left(x_n+e_n\right)-\left\langle\nabla f\left(x_n+e_n\right), x_n-\left(x_n+e_n\right)\right\rangle \\ &= f\left(x_n\right)-f\left(x_n+e_n\right)+\left\langle\nabla f\left(x_n+e_n\right), e_n\right\rangle . \end{aligned}$$

The function f is bounded on bounded subsets of X and therefore ∇f is also bounded on bounded subsets of X (see [**18**, Proposition 1.1.11, p. 16]). In addition, f is uniformly Fréchet differentiable and therefore f is uniformly continuous on bounded subsets (see [**1**, Theorem 1.8, p. 13]). Hence, since $\lim_{n\to\infty} e_n=0$, we have that

$$\lim_{n\to\infty} D_f\left(x_n, x_n+e_n\right)=0. \tag{3.16}$$

The three point identity (see (2.3)) now implies that

$$\begin{aligned} D_f\left(x_{n+1}, x_n+e_n\right) = {} & D_f\left(x_{n+1}, x_n\right)+D_f\left(x_n, x_n+e_n\right) \\ & +\left\langle\nabla f\left(x_n\right)-\nabla f\left(x_n+e_n\right), x_{n+1}-x_n\right\rangle . \end{aligned}$$

Since ∇f is bounded on bounded subsets of X, $\{x_n\}_{n\in\mathbb{N}}$ and $\{x_n+e_n\}_{n\in\mathbb{N}}$ are bounded, $\lim_{n\to\infty}(x_{n+1}-x_n)=0$, (3.15) and (3.16) we get

$$\lim_{n\to\infty} D_f\left(x_{n+1}, x_n+e_n\right)=0. \tag{3.17}$$

Next it follows from the inclusion $x_{n+1}\in C_n$ (in the case of Algorithm (3.5)) or $x_{n+1}\in H_n$ (in the case of Algorithm (3.6)) that

$$D_f\left(x_{n+1}, y_n\right) \leq D_f\left(x_{n+1}, x_n+e_n\right)$$

hence (3.17) leads to $\lim_{n\to\infty} D_f\left(x_{n+1}, y_n\right)=0$. Since $\{y_n\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.3), Proposition 2.4 now implies that

$$\lim_{n\to\infty}\left(y_n-x_{n+1}\right)=0.$$

Therefore

$$\left\|y_n-x_n\right\| \leq\left\|y_n-x_{n+1}\right\|+\left\|x_{n+1}-x_n\right\| \to 0.$$

Since $\lim_{n\to\infty} e_n=0$, it follows that

$$\lim_{n\to\infty}\left\|y_n-\left(x_n+e_n\right)\right\|=0.$$

Since f is uniformly Fréchet differentiable we get from Proposition 2.1 that

$$\lim_{n\to\infty}\left\|\nabla f\left(x_n+e_n\right)-\nabla f\left(y_n\right)\right\|_*=0. \tag{3.18}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous (see [**1**, Theorem 1.8, p. 13]) and therefore

$$\lim_{n\to\infty} |f(y_n) - f(x_n + e_n)| = 0.$$

Hence, from the definition of Bregman distance (see (2.2)), we get that

$$\begin{aligned}(3.19)\qquad &\lim_{n\to\infty} D_f(y_n, x_n + e_n)\\ &= \lim_{n\to\infty} \left[f(y_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), y_n - (x_n + e_n)\rangle\right] = 0.\end{aligned}$$

Let $u \in Z$. From the three point identity (see (2.3)) we obtain that

$$D_f(u, x_n + e_n) - D_f(u, y_n) = D_f(y_n, x_n + e_n) + \langle \nabla f(x_n + e_n) - \nabla f(y_n), y_n - u\rangle.$$

Since the sequence $\{y_n\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.3) we obtain from (3.18) and(3.19) that

$$\lim_{n\to\infty} \left(D_f(u, x_n + e_n) - D_f(u, y_n)\right) = 0. \tag{3.20}$$

Thence from (3.20) we get that

$$\lim_{n\to\infty} \left(D_f(u, x_n + e_n) - D_f\left(u, S_n^N(x_n + e_n)\right)\right) = 0$$

for any $u \in Z$. From (2.9), (2.10), (3.7) and (3.13) we get that

$$\begin{aligned}D_f\left(S_n^i(x_n + e_n), S_n^{i-1}(x_n + e_n)\right) &= D_f\left(T_n^i\left(S_n^{i-1}(x_n + e_n)\right), S_n^{i-1}(x_n + e_n)\right)\\ &\le D_f\left(u, S_n^{i-1}(x_n + e_n)\right) - D_f\left(u, S_n^i(x_n + e_n)\right)\\ &\le D_f(u, x_n + e_n) - D_f(u, y_n).\end{aligned}$$

Hence from (3.20) we get that

$$\lim_{n\to\infty} D_f\left(S_n^i(x_n + e_n), S_n^{i-1}(x_n + e_n)\right) = 0, \tag{3.21}$$

for any $i = 1, \ldots, N$. Therefore from Proposition 2.4 and the fact that $\left\{S_n^i(x_n + e_n)\right\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.3) we obtain that

$$\lim_{n\to\infty} \left(S_n^i(x_n + e_n) - S_n^{i-1}(x_n + e_n)\right) = 0, \tag{3.22}$$

for any $i = 1, \ldots, N$. From the three point identity (see (2.3)) we get that

$$\begin{aligned}&D_f\left(S_n^i(x_n + e_n), x_n + e_n\right) - D_f\left(S_n^{i-1}(x_n + e_n), x_n + e_n\right)\\ &= D_f\left(S_n^i(x_n + e_n), S_n^{i-1}(x_n + e_n)\right)\\ &+ \left\langle \nabla f(x_n + e_n) - \nabla f\left(S_n^{i-1}(x_n + e_n)\right), S_n^{i-1}(x_n + e_n) - S_n^i(x_n + e_n)\right\rangle.\end{aligned}$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.2) and the sequence $\left\{S_n^i(x_n + e_n)\right\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.3). Hence, from (3.21) and (3.22) we get that

$$\lim_{n\to\infty} \left(D_f\left(S_n^i(x_n + e_n), x_n + e_n\right) - D_f\left(S_n^{i-1}(x_n + e_n), x_n + e_n\right)\right) = 0. \tag{3.23}$$

Since

$$\lim_{n\to\infty} D_f\left(y_n, x_n+e_n\right) = \lim_{n\to\infty} D_f\left(S_n^N\left(x_n+e_n\right), x_n+e_n\right) = 0$$

we obtain from (3.23) that

$$\lim_{n\to+\infty} D_f\left(S_n^i\left(x_n+e_n\right), x_n+e_n\right) = 0$$

for any $i = 1, \ldots, N$. Proposition 2.4 and the fact that $\left\{x_n+e_n\right\}_{n\in\mathbb{N}}$ is bounded (see Lemma 3.2) now implies that

$$\lim_{n\to+\infty} \left(S_n^i\left(x_n+e_n\right) - \left(x_n+e_n\right)\right) = 0 \tag{3.24}$$

for any $i = 1, \ldots, N$, that is,

$$\lim_{n\to\infty} \left(\mathrm{Res}_{\lambda_n^i A_i}^f\left(S_n^{i-1}\left(x_n+e_n\right)\right) - \left(x_n+e_n\right)\right) = 0$$

for any $i = 1, \ldots, N$. From the definition of the resolvent (see 2.4) it follows that

$$\nabla f\left(S_n^{i-1}\left(x_n+e_n\right)\right)\right) \in \left(\nabla f + \lambda_n^i A_i\right)\left(S_n^i\left(x_n+e_n\right)\right)$$

hence

$$\xi_n^i := \frac{1}{\lambda_n^i}\left(\nabla f\left(S_n^{i-1}\left(x_n+e_n\right)\right) - \nabla f\left(S_n^i\left(x_n+e_n\right)\right)\right) \in A_i\left(S_n^i\left(x_n+e_n\right)\right) \tag{3.25}$$

for any $i = 1, \ldots, N$. Applying Proposition 2.1 to (3.22) we get that

$$\lim_{n\to\infty} \left\|\nabla f\left(S_n^{i-1}\left(x_n+e_n\right)\right) - \nabla f\left(S_n^i\left(x_n+e_n\right)\right)\right\|_* = 0.$$

Now let $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$ be a weakly convergent subsequence of $\left\{x_n\right\}_{n\in\mathbb{N}}$ and denote its weak limit by v. Then from (3.24) it follows that $\left\{S_{n_k}^i\left(x_{n_k}+e_{n_k}\right)\right\}_{k\in\mathbb{N}}$, $i = 1, \ldots, N$, also converges weakly to v. Since $\liminf_{n\to\infty} \lambda_n^i > 0$, it follows from (3.25) that

$$\lim_{n\to+\infty} \xi_n^i = 0^*$$

for any $i = 1, \ldots, N$. From the monotonicity of A_i it follows that

$$\left\langle \eta - \xi_n^i, z - S_{n_k}^i\left(x_{n_k}+e_{n_k}\right)\right\rangle \geq 0,$$

for all $(z, \eta) \in \mathrm{graph}\left(A_i\right)$ and for all $i = 1, \ldots, N$. This, in turn, implies that

$$\langle \eta, z - v\rangle \geq 0$$

for all $(z, \eta) \in \mathrm{graph}\left(A_i\right)$ for any $i = 1, \ldots, N$. Therefore, using the maximal monotonicity of A_i, we now obtain that $v \in A_i^{-1}\left(0^*\right)$ for each $i = 1, 2, \ldots, N$. Thus $v \in Z$ and this proves the result. □

LEMMA 3.5. *The sequence $\left\{x_n\right\}_{n\in\mathbb{N}}$ which is generated by Algorithm (3.5) or (3.6) converges strongly to $\mathrm{proj}_Z^f\left(x_0\right)$.*

PROOF. In order to prove the result we will use Proposition 2.6. Let $\tilde{u} = \mathrm{proj}_Z^f(x_0)$. In both algorithms we have that $D_f(x_{n+1}, x_0) \leq D_f(\tilde{u}, x_0)$. Indeed, in Algorithm (3.5) we have that $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0)$ and Z is contained in $C_n \bigcap Q_n$. In Algorithm (3.6) we have that $x_{n+1} = \mathrm{proj}^f_{H_{n+1}}(x_0)$ and Z is contained in H_{n+1}.

Therefore Proposition 2.6 implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\tilde{u}$, as claimed. □

4. Equilibrium Problem

Let K be a closed and convex subset of X, and let $g : K \times K \to mathbbR$ be a bifunction satisfying the following conditions [**12, 26**]:

(C1) $g(x, x) = 0$ for all $x \in K$;

(C2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;

(C3) for all $x, y, z \in K$,

$$\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x, y);$$

(C4) for each $x \in K$, $g(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problem corresponding to g is to find $\bar{x} \in K$ such that

$$g(\bar{x}, y) \geq 0 \quad \forall y \in K. \tag{4.1}$$

The solution set of (4.1) is denoted by $EP(g)$.

It is well known that many interesting and complicated problems in nonlinear analysis, such as complementarity, fixed point, Nash equilibria, optimization, saddle point and variational inequality, can be reformulated as equilibrium problem (see, for instance, [**12**]). There are several papers available in the literature which are devoted to this problem. Most of them deal with conditions for the existence of solution (see, for example, [**30, 32**]). However, there are only a few papers that deal with iterative procedures for solving equilibrium problems in finite as well as infinite-dimensional spaces (see, for instance, [**26, 31, 39, 40, 41, 47, 48**]).

The *resolvent of a bifunction* $g : K \times K \to \mathbb{R}$ [**26**] is the operator $\mathrm{Res}_g^f : X \to 2^K$, defined by

$$\mathrm{Res}_g^f(x) = \{z \in K : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \, \forall y \in K\}.$$

A function f is said to be *supercoercive* if $\lim_{\|x\| \to \infty} (f(x) / \|x\|) = +\infty$. Now we list several properties of the resolvent of bifunctions.

PROPOSITION 4.1 (*cf.* [**39**, Lemmas 1 and 2. pp. 130-131]). *Let $f : X \to (-\infty, +\infty]$ be a supercoercive Legendre function. Let K be a closed and convex subset of X. If the bifunction $g : K \times K \to \mathbb{R}$ satisfies conditions (C1)–(C4), then:*

(i) $\mathrm{dom}\,\mathrm{Res}_g^f = X$*;*

(ii) Res_g^f *is single-valued;*

(iii) Res_g^f *is a BFNE operator;*

(iv) the set of fixed points of Res_g^f *is the solution set of the corresponding equilibrium problem, i.e.,* $F\left(\operatorname{Res}_g^f\right) = EP(g)$*;*

(v) $EP(g)$ *is a closed and convex subset of* K*.*

Let $g : K \times K \to \mathbb{R}$ be a bifunction and define the mapping $A_g : X \to 2^{X^*}$ in the following way:

$$A_g(x) := \begin{cases} \{\xi \in X^* : g(x,y) \geq \langle \xi, y - x\rangle \ \forall y \in K\} & , \quad x \in K \\ \varnothing & , \quad x \notin K. \end{cases} \tag{4.2}$$

In the following result we show that under some properties of the function f we can generate maximal monotone operator A_g from the bifunction g.

PROPOSITION 4.2. *Let* $f : X \to (-\infty, +\infty]$ *be a supercoercive, Legendre, Fréchet differentiable and totally convex function. Let* K *be a closed and convex subset of* X *and assume that the bifunction* $g : K \times K \to \mathbb{R}$ *satisfies conditions (C1)–(C4), then:*

(i) $EP(g) = A_g^{-1}(0^*)$*;*

(ii) A_g *is maximal monotone mapping;*

(iii) $\operatorname{Res}_g^f = \operatorname{Res}_{A_g}^f$*.*

PROOF. (i) If $x \in K$ then from the definition of the mapping A_g (see (4.2)) and (4.1) we get that

$$x \in A_g^{-1}(0^*) \Leftrightarrow g(x,y) \geq 0 \, \forall y \in K \Leftrightarrow x \in EP(g).$$

(ii) We first prove that A_g is monotone mapping. Let (x, ξ) and (y, η) belong to the graph of A_g. By definition of the mapping A_g (see (4.2)) we get that

$$g(x,z) \geq \langle \xi, z - x\rangle \text{ and } g(y,z) \geq \langle \eta, z - y\rangle$$

for any $z \in K$. In particular we have that

$$g(x,y) \geq \langle \xi, y - x\rangle \text{ and } g(y,x) \geq \langle \eta, x - y\rangle.$$

From Condition (C2) we obtain that

$$0 \geq g(x,y) + g(y,x) \geq \langle \xi - \eta, y - x\rangle$$

that is $\langle \xi - \eta, x - y\rangle \geq 0$ which means that A_g is monotone mapping (see (2.5)). In order to show that A_g is maximal monotone mapping it is enough to show that $\operatorname{ran}(A_g + \nabla f) = X^*$ (see [**11**, Corollary 2.3, p. 3]). Let $\xi \in X^*$, from [**19**, Proposition 2.3, p. 39] and [**50**, Theorem 3.5.10, p. 164] we get that f is cofinite, that is, $\operatorname{dom} f^* = X^*$ and therefore $\operatorname{ran} \nabla f = \operatorname{int} \operatorname{dom} f^* = X^*$ which means that ∇f is surjective. Then there exists $x \in X$ such that $\nabla f(x) = \xi$. From Proposition

4.1(i) we know that the resolvent of g has full domain and therefore from the definition of Res_g^f we get that

$$g\left(\mathrm{Res}_g^f(x), y\right) + \left\langle \nabla f\left(\mathrm{Res}_g^f(x)\right) - \nabla f(x), y - \mathrm{Res}_g^f(x)\right\rangle \geq 0$$

for any $y \in K$, that is,

$$g\left(\mathrm{Res}_g^f(x), y\right) \geq \left\langle \nabla f(x) - \nabla f\left(\mathrm{Res}_g^f(x)\right), y - \mathrm{Res}_g^f(x)\right\rangle$$

for any $y \in K$. This shows that $\nabla f(x) - \nabla f\left(\mathrm{Res}_g^f(x)\right) \in A_g\left(\mathrm{Res}_g^f(x)\right)$ (see (4.2)). Therefore

$$\xi = \nabla f(x) \in (\nabla f + A_g)\left(\mathrm{Res}_g^f(x)\right) \tag{4.3}$$

which means that $\xi \in \mathrm{ran}\,(A_g + \nabla f)$. This completes the proof.
(iii) As we noted in the Preliminaries (Subsection 2.3) the resolvent, $\mathrm{Res}_{A_g}^f$, of a maximal monotone mapping A_g is single valued. From Proposition 4.1(ii) the resolvent Res_g^f is single valued too. Now we obtain from (4.3) that

$$\mathrm{Res}_{A_g}^f = (A_g + \nabla f)^{-1} \circ \nabla f = \mathrm{Res}_g^f$$

as asserted. □

As we have seen in Proposition 4.1(i) and (iii) the operator $A = \mathrm{Res}_g^f$ is BFNE and with full domain, therefore, from [**11**, Proposition 5.1, p. 7] the mapping $B = \nabla f \circ A^{-1} - \nabla f$ is maximal monotone. This fact also follows from Proposition 4.2(ii) where we proved that A_g is maximal monotone mapping. Therefore $B = A_g$, indeed, from Proposition 4.2(iii)

$$B = \nabla f \circ \left(\mathrm{Res}_g^f\right)^{-1} - \nabla f = \nabla f \circ \left(\mathrm{Res}_{A_g}^f\right)^{-1} - \nabla f = A_g.$$

Based on Algorithms 3.5 and 3.6 we propose two methods for solving system of finite number of equilibrium problems.

Theorem 4.1. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. Let $g_i : K_i \times K_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be N bifunctions that satisfy conditions (C1)–(C4) such that $E := \bigcap_{i=1}^N EP(g_i) \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a supercoercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ which is generated by Algorithms (3.5) or (3.6) with*

$$y_n = \mathrm{Res}_{\lambda_n^N g_N}^f \cdots \mathrm{Res}_{\lambda_n^1 g_1}^f (x_n + e_n)$$

is well defined. If the sequence of errors $\{e_n\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to\infty} e_n = 0$ and for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_E^f(x_0)$ as $n \to \infty$.

5. Convex Feasibility Problems

Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X. The convex feasibility problem (CFP) is to find an element in the assumed nonempty intersection $\bigcap_{i=1}^{N} K_i$ (see [**2**]). It is clear that $F\left(\mathrm{proj}_{K_i}^f\right) = K_i$ for any $i = 1, 2, \ldots, N$. Based on Algorithms 3.5 and 3.6 we propose two methods for solving the convex feasibility problem.

Theorem 5.1. *Let K_i, $i = 1, 2, \ldots, N$, be N nonempty, closed and convex subsets of X such that $K := \bigcap_{i=1}^{N} K_i \neq \varnothing$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Then, for each $x_0 \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ which is generated by Algorithms (3.5) or (3.6) with*

$$y_n = \mathrm{proj}_{\lambda_n^N, K_N}^f \cdots \mathrm{proj}_{\lambda_n^1, K_1}^f (x_n + e_n)$$

is well defined. If the sequence of errors $\{e_n\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to\infty} e_n = 0$ and for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}_K^f (x_0)$ as $n \to \infty$.

6. Particular Cases

6.1. Uniformly Smooth and Uniformly Convex Banach Spaces. In this subsection we assume that X is a uniformly smooth and uniformly convex Banach space. For instance, we assume that $X = \ell^p$ or $X = L^p$ with $p \in (1, +\infty)$ and $f(x) = (1/p)\|x\|^p$. In this case the function f is Legendre (see [**4**, Lemma 6.2, p.24]) and uniformly Fréchet differentiable on bounded subsets of X. According to [**?**, Corollary 1(ii), p. 325], f is sequentially consistent since X is uniformly convex and hence f is totally convex on bounded subsets of X. Therefore our result holds in this setting. This means that our algorithms are more flexible than previous algorithms because they leave us the freedom of fitting the function f to the nature of the mapping A and of the space X in ways which make the application of these algorithms simpler. These computations can be simplified by an appropriate choice of function f than those required in other algorithms, which correspond to $f(x) = (1/2)\|x\|^2$.

6.2. Hilbert Spaces. In this subsection we assume that X is a Hilbert space. We also assume that the function f is equal to $(1/2)\|\cdot\|^2$. It is well known that in this case $X = X^*$ and $\nabla f = I$, where I is the identity operator. Now we list our main notions under these assumptions.

(1) The Bregman distance $D_f(x, y)$ and the Bregman projection proj_K^f become $(1/2)\|x - y\|^2$ and the metric projection P_K, respectively.

(2) The class of BFNE operators become the class of firmly nonexpansive operators: recall that in this setting an operator $T : K \to K$ is called *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$$

for any $x, y \in K$.

(3) The resolvent Res_A^f of a mapping A become the classical resolvent $R_A = (I + A)^{-1}$.

Now our Algorithms (3.5) and (3.6) take the following form:

$$\begin{cases} x_0 \in X, \\ y_n = \left(I + \lambda_n^N A_N\right)^{-1} \cdots \left(I + \lambda_n^1 A_1\right)^{-1} (x_n + e_n), \\ C_n = \left\{z \in X : \|z, y_n\|^2 \le \|z, x_n + e_n\|^2\right\}, \\ Q_n = \{z \in X : \langle x_0 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{6.1}$$

and

$$\begin{cases} x_0 \in X, \\ H_0 = X, \\ y_n = \left(I + \lambda_n^N A_N\right)^{-1} \cdots \left(I + \lambda_n^1 A_1\right)^{-1} (x_n + e_n), \\ H_{n+1} = \left\{z \in H_n : \|z, y_n\|^2 \le \|z, x_n + e_n\|^2\right\}, \\ x_{n+1} = P^f_{H_{n+1}}(x_0), \qquad n = 0, 1, 2, \ldots. \end{cases} \tag{6.2}$$

These algorithms seems to be even new in this setting. For instance, one can find algorithms for finding common zeroes of finitely many maximal monotone mappings in Hilbert space, see [**6**]. In Algorithm (6.1) we should compute the orthogonal projection onto the intersection of two half-spaces. This is a simple task, since the orthogonal projection onto the intersection of two halfspaces (see [**8**, Section 28.3]):

$$T = \{x \in \mathbb{R}^n : \langle a_1, x\rangle \le b_1,\ \langle a_2, x\rangle \le b_2\}\ (a_1, a_2 \in \mathbb{R}^n,\ b_1, b_2 \in \mathbb{R})$$

is given by the following explicit formula:

$$P_T(x) = \begin{cases} x, & \alpha \le 0 \text{ and } \beta \le 0, \\ x - (\beta/\nu)\, a_2, & \alpha \le \pi(\beta/\nu) \text{ and } \beta > 0, \\ x - (\alpha/\mu)\, a_1, & \beta \le \pi(\alpha/\mu) \text{ and } \alpha > 0, \\ x + (\alpha/\rho)(\pi a_2 - \nu a_1) + (\beta/\rho)(\pi a_1 - \mu a_2), & \text{otherwise,} \end{cases}$$

where here

$$\pi = \langle a_1, a_2\rangle,\ \mu = \|a_1\|^2,\ \nu = \|a_2\|^2,\ \rho = \mu\nu - \pi^2$$

and

$$\alpha = \langle a_1, x\rangle - b_1 \text{ and } \beta = \langle a_2, x\rangle - b_2.$$

In our case

$$a_1 = x_n - y_n,\ b_1 = \left(\|x_n\|^2 - \|y_n\|^2\right)/2,\ a_2 = x_0 - x_n,\ b_2 = \langle x_0 - x_n, x_n\rangle .$$

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References

[1] Ambrosetti, A. and Prodi, G.: A Primer of Nonlinear Analysis, *Cambridge University Press*, Cambridge, 1993.

[2] Bauschke, H. H. and Borwein, J. M.: On projection algorithms for solving convex feasibility problems, *SIAM Review* **38**(1996), 367–426.

[3] Bauschke, H. H. and Borwein, J. M.: Legendre functions and the method of random Bregman projections, *J. Convex Anal.* **4** (1997), 27–67.

[4] Bauschke, H. H., Borwein, J. M. and Combettes, P. L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Comm. Contemp. Math.* **3** (2001), 615–647.

[5] Bauschke, H. H., Borwein, J. M. and Combettes, P. L.: Bregman monotone optimization algorithms, *SIAM J. Control Optim.* **42** (2003), 596–636.

[6] Bauschke, H. H. and Combettes, P. L.: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.* **26** (2001), 248–264.

[7] Bauschke, H. H. and Combettes, P. L.: Construction of best Bregman approximations in reflexive Banach spaces, *Proc. Amer. Math. Soc.* **131** (2003), 3757–3766.

[8] Bauschke, H. H. and Combettes, P. L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, *Springer*, 2011.

[9] Bauschke, H. H., Combettes, P. L. and Reich, S.: The asymptotic behavior of the composition of two resolvents, *Nonlinear Anal.* **60** (2005), 283–301.

[10] Bauschke, H. H., Matoušková, E. and Reich, S.: Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal.* **56** (2004), 715–738.

[11] Bauschke, H. H., Wang, X. and Yao, L.: General resolvents for monotone operators: characterization and extension, *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems*, Medical Physics Publishing, Madison, WI, USA, 2010, 57–74.

[12] Blum, E. and Oettli, W.: From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), 123–145.

[13] Bregman, L. M.: A relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming, *USSR Comput. Math. and Math. Phys.* **7** (1967), 200–217.

[14] Brézis, H. and Lions, P.-L.: Produits infinis de résolvantes, *Israel J. Math.* **29** (1978), 329–345.

[15] Browder, F. E.: Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A* **56** (1966), 1080–1086.

[16] Bruck, R. E. and Reich, S.: Nonexpansive projections and resolvents of accretive operators, *Houston J. Math.* **3** (1977), 459–470.

[17] Butnariu, D., Censor, Y. and Reich, S.: Iterative averaging of entropic projections for solving stochastic convex feasibility problems, *Comp. Optim. Appl.* **8** (1997), 21–39.

[18] Butnariu, D. and Iusem, A. N.: Totally convex functions for fixed points computation and infinite dimensional optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.

[19] Butnariu, D., Iusem, A. N. and Zălinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Anal.* **10** (2003), 35–61.

[20] Butnariu, D. and Kassay, G.: A proximal-projection method for finding zeroes of set-valued operators, *SIAM J. Control Optim.* **47** (2008), 2096–2136.

[21] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.* **2006** (2006), Art. ID 84919, 1–39.

[22] Censor, Y. and Lent, A.: An iterative row-action method for interval convex programming, *J. Optim. Theory Appl.* **34** (1981), 321–353.

[23] Censor, Y. and Reich, S.: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, *Optimization* **37** (1996), 323–339.

[24] Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, *Kluwer Academic Publishers*, Netherlands, 1990.

[25] Combettes, P. L.: Strong convergence of block-iterative outer approximation methods for convex optimization, *SIAM J. Control Optim.* **38** (2000), 538–565.

[26] Combettes, P. L. and Hirstoaga, S. A.: Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* **6** (2005), 117–136.

[27] Gárciga Otero, R. and Svaiter, B. F.: A strongly convergent hybrid proximal method in Banach spaces, *J. Math. Anal. Appl.* **289** (2004), 700–711.

[28] Genel, A., Lindenstrauss, J.: An example concerning fixed points, *Israel J. Math.* **22** (1975), 81–86.

[29] Güler, O.: On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.* **29** (1991), 403–419.

[30] Iusem, A.N., Kassay, G. and W. Sosa.: On certain conditions for the existence of solutions of equilibrium problems, *Math. Program.* **116** (2009), 259–273.

[31] Iusem, A.N. and Nasri, M. : Inexact proximal point methods for equilibrium problems in Banach spaces, *Numer. Funct. Anal. Optim.* **28** (2007), 1279–1308.

[32] Iusem, A.N. and W. Sosa.: Iterative algorithms for equilibrium problems, *Optimization* **52** (2003), 301–316.

[33] Martinet, B.: Régularisation d'inéquations variationelles par approximations successives, *Revue Française d'Informatique et de Recherche Opérationelle* **4** (1970), 154–159.

[34] Phelps, R. R.: Convex Functions, Monotone Operators, and Differentiability, Lecture Notes in Mathematics **1364**, 2nd Edition, *Springer Verlag*, Berlin, 1993.

[35] Reich, S.: A weak convergence theorem for the alternating method with Bregman distances, in *Theory and applications of nonlinear operators of accretive and monotone type*, Marcel Dekker, New York, 1996, 313–318.

[36] Reich, S. and Sabach, S.: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces, *J. Nonlinear Convex Anal.* **10** (2009), 471–485.

[37] Reich, S. and Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces, *Numer. Funct. Anal. Optim.* **31** (2010), 22–44.

[38] Reich, S. and Sabach, S.: Existence and approximation of fixed points of Bregman firmly non-expansive mappings in reflexive Banach spaces, *Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer*, New York, 299–314.

[39] Reich, S. and Sabach, S.: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, *Nonlinear Analysis* **73** (2010), 122–135.

[40] Reich, S. and Sabach, S.: A projection method for solving nonlinear problems in reflexive Banach spaces, preprint, *J. Fixed Point Theory Appl.* **9** (2011), 101–116.

[41] Reich, S. and Sabach, S.: Three Strong Convergence Theorems for Iterative Methods for Solving Equilibrium Problems in Reflexive Banach Spaces, Contemporary Mathematics, AMS, to appear.

[42] Reich, S. and Zaslavski, A. J.: Infinite products of resolvents of accretive operator, *Topol. Methods Nonlinear Anal.* **15** (2000), 153–168.

[43] Resmerita, E.: On total convexity, Bregman projections and stability in Banach spaces, *J. Convex Anal.* **11** (2004), 1–16.

[44] Rockafellar, R. T.: Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), 877–898.

[45] Rockafellar, R. T.: Augmented Lagrangians and applications of the proximal point algorithm in convex programming, *Math. Oper. Res.* **1** (1976), 97–116.

[46] Solodov, M. V. and Svaiter, B. F.: Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Program.* **87** (2000), 189–202.

[47] Takahashi, W. and Zembayashi, K.: Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, *Fixed point Theory Appl.* **2008** (2008), Art. ID 528476, 1-11.

[48] Takahashi, W. and Zembayashi, K.: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **70** (2009), 45–57.

[49] Vladimirov, A. A., Nesterov, Y. E. and Chekanov, Y. N.: On uniformly convex functionals (Russian), *Vestnik Moskovskaya Universiteta, Seria Matematika y Kibernetika* **3** (1978), 12–23.

[50] Zălinescu, C.: Convex analysis in general vector spaces, *World Scientific Publishing*, Singapore, 2002.

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